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What should be the best retail strategy to deal with an unequal shipment from an unreliable manufacturer?

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A R T I C L E I N F O

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A B S T R A C T

Unreliability of the manufacturer is a challenging issue for a retailer in order to provide service to consumers and meet the market demand. Due to the unreliability of the manufacturer, the lead time increases, causing shortages. In turn, the retailer faces huge shortages and losses. The lead time can be minimized by reducing the flow time during work-in-process. To reduce the holding cost of the retailer under an increasing demand, the single-setup-multi-unequal-increasing-delivery is introduced by the unreliable manufacturer. But delivered products to the retailer variable demand are lower in volume than the ordered products. Due to the variable demand that is selling price and service dependent, the number of shipments during transportation increases for the single-setup-multi-unequal-increasing-delivery policy. The main goal of this research is to manage unequal shipments from the unreliable manufacturer for gaining more profit. The stochastic optimization approach is considered for the analytical solution. The quasi-closed-form solution is determined for the decision variables of the model. The study is illustrated both numerically and graphically. Results prove that the retailer can still control the profit if the manufacturer can reduce the flow time of the production and maintain a perfect retailing strategy. The research shows that the single-setup-multi-unequal-increasing-delivery policy is 1.14% more profitable than the single-setup-multi-delivery policy, and 8.53% more profitable than the single-setup-single-delivery policy.

1. Introduction

The unreliability issue among players in a supply chain (SC) is critical in running a SC smoothly. Nowadays, some players are opportunistic and hide information about the product, price, delivery time, quality, and service. Companies' good reputations and products' demand are badly affected by this cause. Thus, controlling the problem of unreliable players is essential for ensuring that the SC runs smoothly. In this proposed manufacturer-retailer SC model, the manufacturer is unreliable and hides information from the retailer. After reaching the inventory reorder point, the retailer orders a certain quantity to the manufacturer. Still, due to the unreliability of the manufacturer, the manufacturer delivers less quantity than the ordered quantity. The manufacturer hides the information regarding this lower amount, as well as the delivery time. As an effect, shortages arise and the retailer suffers a lost sale.

The unreliability of the manufacturer increases the lead time (LT) while delivering a lower volume of product then the ordered quantity, both of which cause shortages. Therefore, it is important to think of ways for reduction of LT and hence, reduction of the amount of sales. Ghasemi et al. (2022) reduced the lead time in a SC model. When shortages arise, many authors consider full backlogging or total lost sales. But

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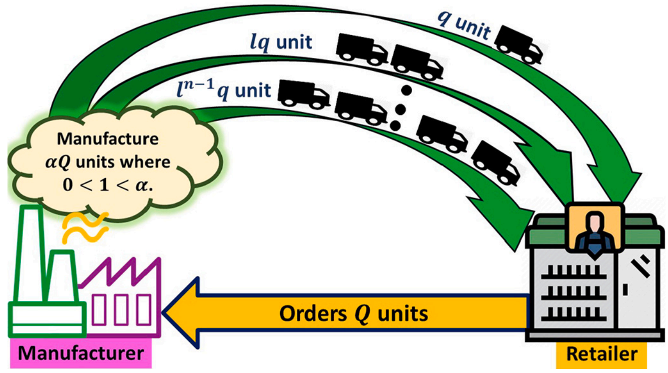

Fig. 1. Single-setup-multiple-unequal-increasing-delivery (SSMUID) policy within a supply chain (SC).

in reality, due to the retailer's good behavior, some customers wait until the product replenishes, especially when the LT is short, which results in partial backlogging. To reduce lost sales, sometimes retailers offer discount. The amount of product kept in hand during the order is called the safety stock. It minimizes lost sales and increases total profit. Nematollahi et al. (2022) considered a single safety stock in their models, but practically, a two-stage safety stock is more beneficial. Again, an LT consists of many parts, such as raw material processing time, manufacturing time, and transportation time. If one of these factors can be controlled, the LT can be reduced.

Transportation plays a vital role in the supply chain model (SCM), and there is a wide range of transportation modes. Many authors considered that the total manufactured product should be transported in a single lot, which is called the single-setup-single-delivery (SSSD) policy (Ahmadi Malakot et al., 2022). Thus, the SSSD policy increased the retailer's holding cost (HC) as well as the total cost of the SC. Therefore, in this type of case, where the HC of the retailer is larger than the HC of the manufacturer, the SSSD policy is not reasonable to use. For these cases, another type of transportation mode, known as single-setup-multi-delivery (SSMD) (Hsiao et al., 2022), may be applied. In this policy, the manufacturer produces the production quantity at a given time but then delivers it to the retailer in different lots. Again, if the lot sizes are unequal, the policy represents the single-setup-multi-unequal-delivery (SSMUD) policy (Iqbal et al., 2022). This study considers a delivery policy in which the lot size increase in multi-orders. This policy is known as single-setup-multi-unequal-increasing-delivery (SSMUID) policy and is described graphically in Fig. 1, which was introduced by Hota et al. (2020).

Market demand is the most important part of an SCM. Generally, the demand is considered to be constant. However, in the real sense, the demand depends on various factors, such as the service of the provider (Hota et al., 2020), selling price of the product (Yadav et al., 2021), advertisement of the product (Gupta, 2022), delivery time (Udayakumar, 2022), product quality, sales volume of the product, and environment (Khorshidvand et al., 2021). Both service level and selling price (SP) dependent demand are examined in this study. To maximize the profit and to minimize the lost sales, a variable backorder price discount which depends on variable backorder rate and two-stage safety stock is applied. In contrast, the production time is minimized in this study. Furthermore, the setup cost (SEC) and ordering cost (OC) can be reduced, while the quality and service level of the product can be increased through different types of investment.

Finally, the profit of an SC with an unreliable manufacturer, single retailer, single type of product, service and SP-dependent demand, two-step safety factors, and raw material processing time reduction is maximized by the distribution-free approach (DFA). The SSMUID policy and container-dependent transportation and carbon emission costs (TCEC) are applied for the transportation.

1.1. Research gaps

Based on the above discussions, following research gaps are found in the literature.

- In literature, several study discussed SCM with reliable players (Malik et al., 2023; Tayyab and Sarkar, 2021). However, the effect of unreliability in retailing industry still a big gap. Thus in this current study, an attempt is taken to fill this gap.
- In the literature, SSSD and SSMD transportation policy are very widely used (Sarkar et al., 2021; Kar et al., 2023). However, SSMUID transportation strategy with unreliable player is still not discussed in existing literature that much. Thus, this study is developed to fill this realistic gap.
- Most of the studies consider either constant demand (Taleizadeh et al., 2023) or selling price-dependent demand (Sarkar et al., 2020). However, selling price and service-dependent demand is limited in the literature. Therefore, a unreliable retailing strategy with selling price and service-dependent variable demand under the consideration of LT reduction is developed in this study.

1.2. Research questions

The following research questions are found in the literature.

- If the manufacturer is unreliable, how does the retailer maintain the profit? If retail products are committed to sending and those are not sent on time, how does the retail industry tackle the market demand situation?
- If the service is increased for the customer, how can the profit be tackled by the retailer? What type of investment is needed for the improved service for the customer?
- What type of price discount should be allowed by the retailer to save from the backorder? What are the possible strategies to maintain the profit regularly despite shortages?

1.3. Contributions in this study

The contributions of this study are described below.

- In this research, one manufacturer (unreliable), one retailer, and one type of product are considered where the demand is related to service and price. If the manufacturer is unreliable, how does the retailer maintain the profit? This study finds the solution to this question.
- For the unreliable manufacturer, shortages rise and the retailer offers a discount (variable) using two-step safety factors to reduce the lost sales. SEC, OC, quality, and service investments are applied, while processing time reduction is involved. The transportation policy is SSMUID, along with container-dependent transportation and CEC. Hence, maintaining profit along with improved service for the customer in the retail industry is dealt with in this study.
- Finally, this study introduces possible strategies to maintain the profit regularly despite shortages. The study proves that the retailer can still control the profit if the manufacturer can reduce the flow time of the production and maintain the perfect retailing strategy.

1.4. Orientation of this study

Section 2 provides a literature review for specific keywords. Section 3 of this study is the problem description, notation, and assumption section. The model formulation is developed in Section 4. The problem and its solution methodologies are explained in Section 5. The result is discussed numerically in Section 6, and the sensitivity analysis is detailed in Section 7. Section 8 contains the managerial insights, and Section 9 explains the model's conclusion and potential future extensions.

2. Literature review

The research is conducted for several area-based reviews.

2.1. Retailer's strategy versus unreliable manufacturer

There are only a few research articles on SCM, which considered unreliability of the manufacturer and the corresponding retailer's strategies. Recently, Hota et al. (2022) studied a single unreliable manufacturer and multiple unreliable suppliers in their model. Gharbi et al. (2022) and Dhahri et al. (2022) reviewed an unreliable manufacturing system within an SCM with multi-retailer. A two-echelon SC with an unreliable manufacturing system was illustrated by Costa et al. (2022). Bouchentouf et al. (2022) explained the vacation policy of an unreliable machine model with multi-station. Previously, Sarkar et al. (2022c) studied an SC for innovative green products but did not ensure the concerned strategy with unequal shipments.

If a large amount of order is received, the manufacturer, even though reliable, may not able to produce those products at a time. The outsourcing strategy is one of the best strategies to maintain the order, even if out-of-range or out of the capacity (Bachar et al., 2023). But if the manufacturer himself is unreliable, then it is very difficult for the retailer to maintain the service for consumers.

2.2. Retail management in supply chain (SC)

A rope ties the manufacturer-retailer relationship, namely, SCM, by which the smooth running of the SC, conceptualization, and modeling are conducted. Several authors extended SCM by introducing different and recent trends. Recently, Yuan and Xiao (2022) studied an SCM with the decoy strategy of the retailer. Jena and Meena (2022) elaborated an omnichannel SCM under return policy and competition in price. Garai and Sarkar (2022) discussed a closed-loop customer-centric SCM for waste reduction with remanufacturing. Tayyab et al. (2020) elaborated a pricing SCM of the retailer with trade-credit strategy and preservation policy, whereas Sarkar et al. (2019) established the impact of using returnable transport items (RTI) within a closed-loop SC.

A tourism SCM was studied by Ma et al. (2021) with the experience of green tourism, whereas Ma (2021) studied a decentralized SC with generic optimization, surplus, and brand advertising. Zhang et al. (2020) studied a dual-channel SC with multiple competing players and multi-demand markets where manufacturers and retailers provide consumer services. Sarkar and Bhuniya (2022) considered a green investment for emission reduction within an SC model, while Ullah and Sarkar (2020) applied RFID in their model to run SCM smoothly and Ahmed et al. (2021) introduced a reworking policy within a global SCM. The players of SCM in the models mentioned above were honest and reliable. But in today's market, players of SCM may be unreliable and hide several information, such as price, demand, transaction information, service, and LT, from other players of that SCM. In this study, the manufacturer of the SCM is unreliable and hides the true quantity of delivered product from the retailer, which causes shortages and results in the retailer losing the goodwill of his customers. Therefore, the market demand for the product may decrease, and the company may lose its reputation. A solution to this problem is described in this proposed study.

2.3. Selling price-dependent demand in retail industry

Generally, the market demand is assumed to be constant; however, this is not realistic. Recently, in a model, Torkaman et al. (2022) considered that demand depends upon the product's price. Malleeswaran and Uthayakumar (2022) discussed an SCM with a discount on price-demand through service level. Mahata and Debnath (2022) illustrated a single-item inventory model under flexibility through preservation and selling price-based demand. Sathishkumar and Iswarya (2022) considered selling price-dependent demand in an SCM with salvage and shortage cost. Saha et al. (2023) studied advertise-dependent market demand in an SC model with a backorder and variable LT. Several authors worked on this issue, but none considered an SCM with an unreliable manufacturer in which demand depends on selling price and service. This lack is fulfilled in this study.

2.4. Processing time reduction in retail management

In traditional SC models, the presumed LT is constant. Still, it is related to many factors, like the quantity ordered, learning factors, raw material processing time, manufacturing time, and transportation time. The unreliability of the manufacturer impacts the LT, which is an important part of

Table 1
Contribution of various authors.

Author(s)	Unreliable	TCEC depends on	Shipment strategy	Safety stock type	Reduction	Demand depends on	Methodology
Mridha et al. (2023a)	information	NC	SSSD	NC	NC	NC	DF
Mishra et al. (2021)	NC	system	NC	NC	emission	SP	CL
Ullah and Sarkar (2020)	NC	NC	NC	NC	NC	NC	CL
Ullah et al. (2021)	NC	RTI	NC	NC	NC	stoc	CL
Habib et al. (2021)	NC	fuel	SSSD	NC	waste	constant	fuzzy
Chen et al. (2021)	channel	NC	NC	NC	NC	SP	MT
Ahmed et al. (2021)	NC	NC	SSSD	NC	NC	NC	CL
Kugele and Sarkar (2023)	system	NC	SSSD	NC	NC	advt	CL
Dey et al. (2021b)	NC	SSMD	SSMD	NC	SEC	SP, service	CL
Sarkar and Guchhait (2023)	retailer	NC	SSSD	NC	CE	NC	CL
This Model	manufacturer	container	SSMUD	two stage	SEC, OC, PT	SP, service	DF

NC - Not considered, SEC - Setup cost, OC - Ordering cost, PT - Production time, ST - setup and transportation time, SP - selling price, BEM - Bayesian equilibrium method, TECC - transportation, and carbon emission cost, CE - carbon emission cost, WD - Weibull distribution, ND - normal distribution, PD - Poisson distribution, CL - classical optimization technique, DF - distribution-free, MT - Monte Carlo test and probability, stoc - Stochastic, RTI - Returnable transport item.

the SCM. A long LT creates a shortage, which increases lost sales. Safety stock and a reduction in LT are two important factors for reducing lost sales.

The concept of decomposing LT into various linear parts with piece-wise different continuous linear crashing costs, each of which may be decreased by taking the LT demand as a normal distribution, was studied by Dey et al. (2021a). Barman and Mahata (2022) reduced the LT in an inventory model. Mousavi and Gholami-borujeni (2021) reduced processing time in their study to reduce pollution. But none of the authors thought about reducing the retailer's HC by applying the new shipment strategy, which is introduced in this study.

2.5. SSMUID policy for retailing

The HC of the retailer and manufacturer are important parts of an SCM. Usually, a player's HC depends on the player's location. Manufacturing houses are generally located in rural areas, while retailing stores and warehouses tend to be located in towns, semi-towns, and metro cities. Naturally, a retailer's HC is higher than a manufacturer's HC. One necessary thing to minimize the HC of the retailer is the policy taken by the manufacturer to deliver the ordered quantity to the retailer. In initial models, no consideration was given to the delivery policy. Then, authors considered the SSSD policy in which the total manufactured quantity is delivered in a single lot. Hsiao et al. (2022) discussed production models with a single shipment strategy. That transportation policy increased the total HC.

Das Roy and Sana (2021) illustrated the SSMD policy, which reduced the total HC of the SC. As demand for a product is not fixed, except for considering equal lot sizes, unequal lot sized consideration is more appropriate. Iqbal et al. (2022) considered unequal and variable lot sizes in his model and named the policy a SSMUD policy. However, none of the authors considered the case of increasing demand, although there are some products whose demand never decreases. For those products, the lot sizes must be increased in every lot. Ahmed and Sarkar (2019) applied a triple bottom line approach for the production of next-generation biofuel and transport them in their respective SC models. But, they used SSSD policy for transportation.

Hota et al. (2020) introduced the SSMUID policy for the products with increasing demand. In this policy, the lot size of each lot increased by a multiple of the previous lot. None of the researchers considered SSMUID policy in an SCM with an unreliable manufacturer and with two-step safety factors. This research aims to cover the lack the by considering two-stage safety factors and processing time reduction in an SCM with an unreliable manufacturer and SSMUID policy.

2.6. Carbon emission in retail industry

But SSMUID policy increased the number of transportation in SC which affects the environment by emitting huge amounts of carbon. Due to the increasing levels of pollution and to save the environment, carbon emission cost (CEC) must be considered in addition to transportation costs. Sun and Zhong (2023) studied the behavior of reducing a two-echelon SC with low-carbon and increasing green marketing. Recently Tiwari et al. (2019) and Sarkar and Sarkar (2020) considered CEC and waste management, respectively, in their studies and discussed methods for reducing it and saving the environment. Sarkar et al. (2022b) studied the combined effect of improvement of production quality and carbon emission from transportation within a sustainable SC model. A low-carbon SCM model of online shopping with an online-to-offline (O2O) policy was reviewed by Wu et al. (2021). However, none of these authors considered CEC within the SCM with an unreliable manufacturer and SSMUID policy. This study fulfills the research gap. In addition to a fixed CEC, a container-dependent CEC is applied to prevent global warming and save the world.

Table 1 gives some previously done work on this field.

3. Problem description, notation, and assumption

Here, the problem description, along with symbols and assumptions, is stated as follows.

3.1. Problem description

Reliability and cost-effective retail strategies are crucial in efficient retail management. The retailer needs to take up some policies or strategies to deal with an unreliable manufacturer which can be the most beneficial for the retail industry.

This study considers a two-echelon SC in which the manufacturer is unreliable with the selling price and service-dependent demand. The manufacturer produces a fraction of the order quantity and hides this information from the retailer. Therefore, shortages and a backordering problem arise. To reduce the backordering status, the retailer offers a backorder price discount and uses a two-stage safety stock.

The LT has components such as setup, transportation, and processing time. The setup time comprises three components: raw material processing, manufacturing, and packing. The manufacturer reduces the raw material processing and manufacturing time by applying a crashing cost. After the manufacturing process, the manufacturer delivers the manufactured quantity to the retailer through several lots to minimize the retailer's HC. The lot sizes increases in multiples of the first lot through a geometric progression series under increasing demand. Investments for SEC reduction, OC reduction, quality upgradation, and service level improvement are used. Finally, the SC total profit is calculated, and from the numerical analysis, several conclusions have been derived.

3.2. Notation

The following symbols are frequently considered to describe the model.

	Decision variables
S	selling price (\$/unit)
ρ	service (in %)
I	investment for setup cost reduction of the manufacturer per batch (\$/batch)
ϕ	probability of out-of-order state
q	first lot size quantity (unit/cycle)
l	shipment's increasing rate
n	shipment number (/cycle)
k	safety factor of the first batch (units)
A	retailer's investment for reducing OC (\$/batch)
π_x	price discount for reducing backorder (\$/unit)
	Dependent variables
k_1	safety factor of batch other then batch 1 (units)
L	replenishment lead time (day)
r	retailer's reorder point (units)
Q	ordered quantity (units)
t_m	manufacturer packing time (day)
	Parameters
A_0	initial ordering cost of the retailer (\$/batch)
κ	scaling parameter of the discrete investment function
α	random yield of the manufacturer (in %)
ξ_1, ξ_3	scaling parameter of service level and price, respectively
ξ_2	shape parameter of service level
D	demand rate (unit/time unit)
P	production rate (unit/time unit)
V_0	manufacturer fixed initial SEC (\$/batch)
μ	shape parameter of the discrete investment function
s_d	manufacturer defective cost (\$/unit)
H_m	manufacturer holding cost (\$/unit/unit time)
ϕ_0	initial probability of the out-of-order state
C_{ftm}	manufacturer fixed transportation cost (\$/unit)
C_{vtm}	manufacturer transportation cost (\$/container)
C_{fcm}	manufacturer fixed carbon emission cost (\$/unit)
C_{vcm}	manufacturer variable carbon emission cost (\$/container)
t_r	raw material processing and manufacturing time of manufacturer (day)
t_m	manufacturing and packing time (day)
γ	capacity of the container (unit)
a_i	i^{th} component of LT with minimum duration (days)
b_i	i^{th} component of LT with normal duration (days)
m_i	crashing cost of the i^{th} component of LT (\$/day)
C_M	manufacturer's unit production cost (\$/unit)
η	shape parameter of the investment function for the service improvement
C_W	retailer's wholesale price (\$/unit)
H_r	retailer's holding cost (\$/unit)
X_1	LT demand of the first batch
X_2	LT demand of batches other than first batch
π_0	marginal profit (\$/unit)
β	ratio of backorder, $0 < \beta < 1$
β_0	upper bound of the backorder ratio
S_{max}	maximum selling price (\$/unit)
S_{min}	minimum selling price (\$/unit)

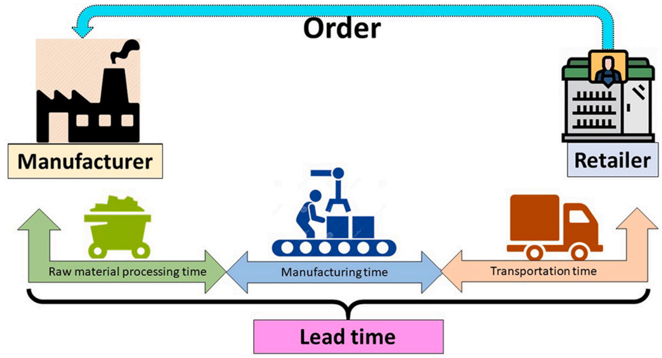

Fig. 2. Processing time reduction.

3.3. Assumptions

1. The demand is dependent on the service level ρ (Dey et al., 2021b) and selling price S, i.e., $D(\rho, S) = \xi_1 \rho^{\xi_2} + \xi_3 \frac{S_{max} - S}{S - S_{min}}$. For the reduction of SEC, a discrete type of investment I is considered here, and the new SEC becomes $V_0 e^{-\mu I}$, where μ is a shape parameter. For the reduction of OC, another discrete type of investment A is considered, and the OC becomes $A_0 e^{-\kappa A}$, where κ is a shape parameter.
2. An order quantity Q is placed at the reorder point r to fill the inventory. After getting the order, the ordered amount Q is produced by the manufacturer. Still, due to the unreliability of the manufacturer, a fraction αQ of the ordered quantity Q is delivered to the retailer, where $0 < \alpha < 1$. To reduce the retailer's HC, the manufacturer ships the amount αQ in n shipments, where the lot sizes of each shipment are unequal. As described in Fig. 1, the manufacturer transports the first lot of size q units, second lot of size ql, third lot of size ql^2, ..., n^{th} lot of size ql^{n-1} (Hota et al., 2022). Therefore, the size of the production batch to the retailer from the manufacturer is as follows:

$$q + ql + ql^2 + \ldots + ql^{n-1} = q\left(\frac{l^n - 1}{l - 1}\right).$$

As the total delivered quantity is αQ,

$$q\left(\frac{l^n - 1}{l - 1}\right) = \alpha Q$$

then, the number of production cycles is $\frac{D}{\alpha Q}$, that is, $\frac{\left(\xi_1 \rho^{\xi_2} + \xi_3 \frac{S_{max} - S}{S - S_{min}}\right)(l-1)}{q(l^n - 1)}$.
3. The LT will be divided into three parts: (i) raw material processing time, (ii) manufacturing time, and (iii) transportation time (Fig. 2). The raw material processing time consists of n mutual components with a_i (with a minimum duration) and b_i (with a normal distribution). $t_r, t_{r_{max}}$ are the raw material processing time and its maximum value, respectively, while the processing time reduction cost is C_i, $i = 1, 2, \ldots, n$, where:

$$\sum_{i=1}^{n} b_i \le t_{r_j} \le \sum_{i=1}^{n} a_i = t_{r_{max}}.$$

It indicates that the components $t_{r,j}, j = 1, 2, \ldots, n$ of the setup time are crashed to their minimum duration are given by,

$$t_{r_j} = \sum_{i=j+1}^{n} a_i - \sum_{i=1}^{j} b_i.$$

The processing time reduction cost is (Dey et al., 2021a)

$$C_j\left(t_{r_{j-1}} - t_{r_j}\right) + \sum_{i=1}^{j+1} C_i\left(a_i - b_i\right). \tag{1}$$

4. An investment $I_\phi(\phi)$ is made to improve the produced defective items during the out-of-order state. A huge loss is caused by an increase in lost sales, which may cause a long LT. For that reason, based on Chauhan et al. (2023) use of a double safety factor is beneficial rather than a single safety factor. Let X_1 be a random variable corresponding to the LT demand x_1 of the first batch during the LT, $L(P, \alpha Q)$ which depends on transportation time (t_s) and setup, as well as the processing time ($\frac{\alpha Q}{P}$), i.e., $L(P, \alpha Q) = t_r + \frac{\alpha Q}{P}$ = raw material processing and manufacturing time + packing time.
The random variable X_1 follows an unknown distribution, with a known mean $DL(P, \alpha Q)$ and a standard deviation of σ. The LT of the first shipment is proportional to the lotsize produced by the manufacturer. Besides, X_2 is a random variable corresponding to the LT demand x_2 of the remaining batches during the LT, $L(t_m)$, which depends only on the raw material and processing and manufacturing time t_m which follows an unknown distribution with a known mean $DL(t_m)$ and a standard deviation σ.
5. The safety stock for the first batch is $S = k\sigma\sqrt{L(P, \alpha Q)} = k\sigma\sqrt{t_r + \frac{\alpha Q}{P}}$, while the safety stock for the second batch onward is defined as $S = k_1 \sigma \sqrt{L(t_m)} = k_1 \sigma \sqrt{t_m}$, which gives the relation between the safety factor as $k_1 = k\sqrt{\frac{t_r + \frac{\alpha Q}{P}}{t_m}}$ for batches $2, 3, \ldots, n$.
6. For shipments $2, 3, \ldots, n$, only raw material processing and manufacturing time t_m are considered for the calculation of LT, and it is assumed that $t_m = \varpi t_r$, where ϖ is the fraction of t_m and t_r.

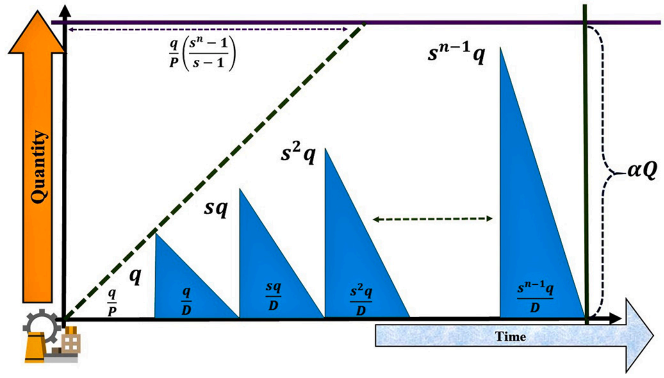

Fig. 3. Inventory position of the retailer.

7. The backorder quantity for the first and second, and onward batches is given by Chauhan et al. (2023) as

$$E(x_1 - R_1)^+ \leq \frac{\sigma}{2}\sqrt{t_r + \frac{\alpha Q}{P}}\left[\sqrt{1+k^2} - k\right] \text{ and } E(x_2 - R_2)^+ \leq \frac{\sigma}{2}\left[\sqrt{t_m + k^2\left(t_r + \frac{\alpha Q}{P}\right)} - k\sqrt{t_r + \frac{\alpha Q}{P}}\right].$$

4. Mathematical model formulation

The SSMUID policy is studied in a manufacturer–retailer SC model with the unreliable manufacturer (the manufacturer does not send the exact ordered amount). Suppose the retailer's ordered volume is Q. In that case, the manufacturer produces the exact volume Q but delivers to the retailer a fraction $\alpha Q, 0 < \alpha < 1$ of the ordered volume, and the quantity αQ is delivered in n (decision variable) lots. The next order is occurred at reorder point r (decision variable), with the OC investment A (decision variable). Then, the unreliable manufacturer starts manufacturing αQ quantity with a production rate P, and the quantity delivers in n shipments. The initial SEC of the manufacturer is V_0, with the variable LT L (decision variable).

The lot size of the first shipment is q (decision variable), and the size of each lot is a multiple of the previous lot by a multiplier $l > 1$ (decision variable). Therefore, the lot size of the second shipment is lq, the third shipment is ql^2, and in this manner, the volume transferred to the retailer on delivery n is ql^{n-1}, $n > 1$. The manufacturer invests an amount I (decision variable) for reducing the SEC and sends the lots of each batch with some costs of transportation C_{ftm} (fixed cost) and C_{vtm} (variable cost). The manufacturer continues the whole delivery of products in n shipments. Throughout the unequal delivery of goods, the manufacturer pays some costs C_{fcm} (fixed cost) and C_{vcm} (variable cost) for carbon emission.

Two-stage safety factors are applied where the safety factor for the first batch is k, and the safety factors of the remaining batches (batch 2, 3, ..., n) are k_1. Hence by Assumption 5, the safety stock of the retailer is $k\sigma\sqrt{t_r + \frac{\alpha Q}{P}}$. The retailer faces shortages for the unreliability of the manufacturer and lost sales. But some customers wait for the product. The ratio of backorder is a variable $\beta, 0 < \beta < 1$ with the upper bound β_0. To minimize the lost sales, the retailer offers a variable price discount of π_x (decision variable) with a marginal profit π_0, which depends on the variable backorder ratio. The manufacturer's selling price is C_W, and the retailer's selling price is S (decision variable), where the maximum and minimum selling price of the product are S_{max} and S_{min}, respectively. The SC total profit is maximized by taking the LT demand with an unknown distribution function.

4.1. Mathematical model for the retailer

Fig. 3 represents the retailer's inventory level. The manufacturer sends the order quantity n times for each cycle of production with the lot sizes $q, ql, ..., ql^{n-1}$, and the transported production batch from the manufacturer to the retailer is

$$q + ql + ... + ql^{n-1} = q\left(\frac{l^n - 1}{l - 1}\right).$$

Thus, the production cycles are $\frac{D}{q\left(\frac{l^n-1}{l-1}\right)}$, i.e.,

$$\frac{\left(\xi_1\rho^{\xi_2} + \xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q(l^n-1)}.$$

4.1.1. Ordering cost (OC) with the investment for reduction

The retailer applies some discrete investment \$$A$ per batch for reducing the OC. Then the new OC of the retailer is $A_0e^{-\kappa A}$, where κ is a shape parameter. Since the production cycle is $\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q(l^n-1)}$, the total amount of OC of the retailer with the investment is as follows:

$$\frac{\left(\xi_1\rho^{\xi_2} + \xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q(l^n-1)}\left(A_0e^{-\kappa A} + A\right). \tag{2}$$

4.1.2. Processing time reduction of the retailer

As described in Assumption 3, in order to reduce the LT, the retailer invests some amount to reduce the processing time. To calculate the amount of investment, the raw material processing is divided into n mutual components of minimum duration $a_1, a_2, \ldots, a_n$ and n mutual components of normal duration $b_1, b_2, \ldots, b_n$. The cost used by the retailer to reduce the processing time (Assumption 3) and improve customers' satisfaction is

$$\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left\{C_j\left(t_{r_{j-1}}-t_{r_j}\right)+\sum_{i=1}^{j+1}C_i\left(a_i-b_i\right)\right\}, \tag{3}$$

where $C_j, j=1,2,\dots,n$, is the cost of processing time reduction and t_r is the raw material processing time, $t_{r_{max}}$ is the maximum value of the raw material processing time and $t_{r_j}=\sum_{i=j+1}^{n}a_i-\sum_{i=1}^{j}b_i$ is the j^{th} component of t_r.

4.1.3. Backorder cost of the retailer

As the manufacturer sends less quantity than the ordered quality, as a result, a shortage must arise. Shortage arises due to unreliability of the manufacturer. When a shortage arises, some customers choose the next retailer to fulfill their demand but some wait due to the good behavior of the retailer. Thus, there is partial backlogging. To prevent the shortage, the retailer considers a two-stage safety stock (Assumption 5). Therefore, the expected annual backorder cost of the retailer per cycle is

$$\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left[\pi_x\beta+\pi_0(1-\beta)\right]\left[E\left(x_1-R_1\right)^++(n-1)E\left(x_1-R_2\right)^+\right], \tag{4}$$

where

$$\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left[\pi_0-\beta_0\pi_x+\frac{\beta_0(\pi_x)^2}{\pi_0}\right]\frac{\sigma}{2}\left[\sqrt{t_r+\frac{\alpha Q}{P}}\left(\sqrt{1+k^2}-k\right)+(n-1)\left(\sqrt{t_m+k^2\left(t_r+\frac{\alpha Q}{P}\right)}-k\sqrt{t_r+\frac{\alpha Q}{P}}\right)\right]. \tag{5}$$

Here, $\beta=\frac{\beta_0\pi_x}{\pi_0}, 0\le\beta_0<1, 0\le\pi_x\le\pi_0$.

4.1.4. Holding cost (HC) of the retailer

The total number of items for each production cycle is obtained by the area of the triangle given in 3, that is

$$\frac{1}{2}q\frac{q}{D}+\frac{1}{2}lq\frac{lq}{D}+\frac{1}{2}l^2q\frac{l^2q}{D}+\dots+\frac{1}{2}l^{n-1}q\frac{l^{n-1}q}{D}=\frac{1}{2}\frac{q^2}{D}\frac{l^{2n}-1}{l^2-1}.$$

Total number of holding items = total number of items in a cycle ∗ cycle length. The holding cost expression is as follows:

$$\left[\frac{1}{2}\frac{q^2}{D}\frac{l^{2n}-1}{l^2-1}\right]\left[\frac{D(l-1)}{q\left(l^n-1\right)}\right]=\frac{q\left(l^n+1\right)}{2(l+1)}.$$

The total HC for the retailer is

$$\left[\frac{q\left(l^n+1\right)}{2(l+1)}+\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)\frac{\sigma}{2}\left\{\sqrt{t_r+\frac{\alpha Q}{P}}\left(\sqrt{1+k^2}-k\right)+(n-1)\left(\sqrt{t_m+k^2\left(t_r+\frac{\alpha Q}{P}\right)}-k\sqrt{t_r+\frac{\alpha Q}{P}}\right)\right\}\right]H_r. \tag{6}$$

4.1.5. Total cost (TC) of the retailer

From equations (2), (3), (5), and (6), the total cost (TC) of the retailer is as follows:

$$\begin{aligned}ATC_R\left(S,\rho,q,l,A,\pi_x,k\right)=&\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left(A_0e^{-\kappa A}+A\right)+\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left\{C_j\left(t_{r,j-1}-t_s\right)+\sum_{i=1}^{j+1}C_i\left(a_i-b_i\right)\right\}\\&+\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left[\pi_0-\beta_0\pi_x+\frac{\beta_0(\pi_x)^2}{\pi_0}\right]\frac{\sigma}{2}\left[\sqrt{t_r+\frac{\alpha Q}{P}}(\sqrt{1+k^2}-k)+(n-1)\sqrt{t_m}\right.\\&\left.\left(\sqrt{1+k^2\frac{t_r+\frac{\alpha Q}{P}}{t_m}}-k\sqrt{\frac{t_r+\frac{\alpha Q}{P}}{t_m}}\right)\right]+\left[\frac{q\left(l^n+1\right)}{2(l+1)}+\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)\frac{\sigma}{2}\left\{\sqrt{t_r+\frac{\alpha Q}{P}}\left(\sqrt{1+k^2}-k\right)\right.\right.\\&\left.\left.+(n-1)\left(\sqrt{t_m+k^2\left(t_r+\frac{\alpha Q}{P}\right)}-k\sqrt{t_r+\frac{\alpha Q}{P}}\right)\right\}\right]H_r.\end{aligned} \tag{7}$$

Here, $0\le\beta_0<1$, $0\le\pi_x\le\pi_0$.

4.1.6. Revenue of the retailer

Since C_W is the wholesale price, S is the SP, and D is the demand for the product. Therefore, the revenue of the retailer is

$$\left(S-C_W\right)\left[\xi_1\rho^{\xi_2}+\xi_3\left(\frac{S_{max}-S}{S-S_{min}}\right)\right]. \tag{8}$$

4.1.7. Total profit of the retailer

Therefore, from equations (7) and (8), the retailer's total profit is as follows:

$$ATP_R\left(S,\rho,q,l,A,\pi_x,k\right)=\left(S-C_W\right)\left[\xi_1\rho^{\xi_2}+\xi_3\left(\frac{S_{max}-S}{S-S_{min}}\right)\right]-\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left(A_0e^{-\kappa A}+A\right)-\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}$$

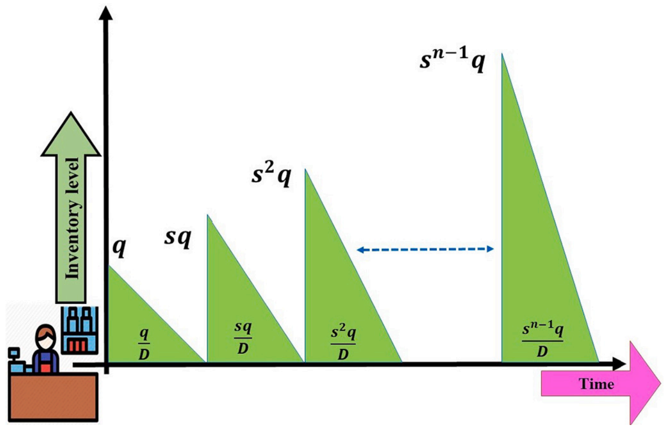

Fig. 4. Figure of the manufacturer.

$$\left\{C_j\left(t_{r,j-1}-t_s\right)+\sum_{i=1}^{j+1}C_i\left(a_i-b_i\right)\right\}-\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left[\pi_0-\beta_0\pi_x+\frac{\beta_0(\pi_x)^2}{\pi_0}\right]\frac{\sigma}{2}\left[\sqrt{t_r+\frac{\alpha Q}{P}}\left(\sqrt{1+k^2}-k\right)\right.$$
$$\left.+(n-1)\left(\sqrt{t_m+k^2\left(t_r+\frac{\alpha Q}{P}\right)}-k\sqrt{t_r+\frac{\alpha Q}{P}}\right)\right]-\left[\frac{q\left(l^n+1\right)}{2(l+1)}+\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)\frac{\sigma}{2}\left\{\sqrt{t_r+\frac{\alpha Q}{P}}\left(\sqrt{1+k^2}-k\right)\right.\right.$$
$$\left.\left.+(n-1)\left(\sqrt{t_m+k^2\left(t_r+\frac{\alpha Q}{P}\right)}-k\sqrt{t_r+\frac{\alpha Q}{P}}\right)\right\}\right]H_r. \tag{9}$$

Here, $0\le\beta_0<1$, $0\le\pi_x\le\pi_0$.

4.2. Mathematical model for the manufacturer

From Fig. 4, the total produced amount is $\alpha Q=q\left(\frac{l^n-1}{l-1}\right)$ and the demand is D. Therefore, the number of production cycle is

$$\frac{D}{\alpha Q}=\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q(l^n-1)}. \tag{10}$$

4.2.1. Setup cost (SEC) with investment for reduction

From Equation (10), the number of production cycle is $\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q(l^n-1)}$ and according to Assumption 1, the initial SEC is V_0, the amount of investment for SEC reduction per batch is I. Therefore, the manufacturer's total SEC along with the discrete investment for reduction of SEC is as follows:

$$\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left(V_0e^{-\mu I}+I\right), \tag{11}$$

where, μ is a known parameter.

4.2.2. Holding cost (HC) of the manufacturer

The area of the rectangle OABD in Fig. 4 is

$$\left(\frac{q}{P}+\frac{q}{D}+\frac{sq}{D}+\ldots+\frac{s^{n-1}q}{D}\right)(\alpha Q)=q^2\left(\frac{1}{P}+\frac{1}{D}\frac{l^n-1}{l-1}\right)\left(\frac{l^n-1}{l-1}\right).$$

The area of the triangle OCD in Fig. 4 is

$$\frac{1}{2}(\alpha Q)\left(\frac{q}{P}\frac{l^n-1}{l-1}\right)=\frac{1}{2}\frac{q^2}{P}\left(\frac{l^n-1}{l-1}\right)^2.$$

The total area of the small triangles in Fig. 4 is

$$\frac{1}{2}q\frac{q}{D}+\frac{1}{2}sq\frac{sq}{D}+\ldots+\frac{1}{2}s^{n-1}q\frac{s^{n-1}q}{D}=\frac{1}{2}\frac{q^2}{D}\left(\frac{s^{2n}-1}{s^2-1}\right).$$

From Fig. 4, manufacturer's total inventory is as follows:

$$q^2\left(\frac{l^n-1}{l-1}\right)\left[\frac{1}{P}+\left(\frac{1}{D}-\frac{1}{2P}\right)\left(\frac{l^n-1}{l-1}\right)-\frac{1}{2D}\left(\frac{l^n+1}{l+1}\right)\right].$$

Thus, the inventory (average) of the manufacturer is

$$q\left[\frac{D}{P}+\left(1-\frac{D}{2P}\right)\left(\frac{l^n-1}{l-1}\right)-\frac{1}{2}\left(\frac{l^n+1}{l+1}\right)\right].$$

The total HC of the manufacturer is

$$q\left[\frac{\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}}{P}+\left(1-\frac{\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}}{2P}\right)\left(\frac{l^n-1}{l-1}\right)-\frac{1}{2}\left(\frac{l^n+1}{l+1}\right)\right]H_m. \tag{12}$$

4.2.3. Defective cost of the manufacturer

The symbol ϕ (very small in general and close to zero) denotes the probability that the production process may reach an out-of-order state during production. At the same time, the cost for a defective unit is s_d. Then, the expected defective cost (annual) will be $s_d\frac{D}{\alpha Q}\frac{\phi(\alpha Q)^2}{2}$, that is $\frac{s_d D\phi(\alpha Q)}{2}$, that is,

$$\frac{s_d\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)\phi q\left(l^n-1\right)}{2(l-1)}. \tag{13}$$

4.2.4. Capital investment cost of the manufacture

To reduce the imperfect quality products, there should be in-control situation. For this purpose, additional investment is necessary to reduce the out-of-order production state. For this, a capital investment $I_\phi(\phi)$ is considered for quality improvement strategy (by reducing the out-of-order stage) as $I_\phi(\phi)=a\log\left(\frac{\phi_0}{\phi}\right)$ for $0<\phi\le\phi_0$. Thus, the annual cost is

$$a\log\left(\frac{\phi_0}{\phi}\right). \tag{14}$$

4.2.5. Transportation and carbon emission cost (TCEC) of the manufacturer

There are two types of transportation cost for the manufacturer, one is fixed and another is container capacity-dependent. If C_{ftm} and C_{fcm} are fixed TCECs per lot for the manufacturer, then the total fixed TCEC for n lots of the manufacturer are $n\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q(l^n-1)}C_{ftm}$ and $n\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q(l^n-1)}C_{fcm}$, respectively. Again, if γ is the capacity of the container, which is assumed to be fixed, C_{ftm} and C_{fcm} are variable transportation and carbon emission costs for the manufacturer, then the manufacturer's variable TCEC per cycle are $\frac{q}{\gamma}\left(\frac{l^n-1}{l-1}\right)C_{vtm}$ and v$\frac{q}{\gamma}\left(\frac{l^n-1}{l-1}\right)C_{vcm}$, respectively.

Therefore, the total TCEC of the manufacturer is as follows:

$$n\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left(C_{ftm}+C_{fcm}\right)+\frac{q}{\gamma}\left(\frac{l^n-1}{l-1}\right)\left(C_{vtm}+C_{vcm}\right). \tag{15}$$

4.2.6. Investment cost for quality issue

An investment for the quality issue is applied, which depends on the service level. The cost applied is

$$\frac{\eta}{2}\rho^2. \tag{16}$$

Here, η is the scaling parameter.

4.2.7. Total cost of the manufacturer

From equations (11), (12), (13), (14), (15), and (16) the TC of the manufacturer is as follows:

$$\begin{aligned}ATC_M(S,\rho,q,l,I,\phi,n)=&\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q(l^n-1)}\left(V_0e^{-\mu I}+I\right)+q\left\{\frac{\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}}{P}+\left(1-\frac{\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}}{2P}\right)\left(\frac{l^n-1}{l-1}\right)-\frac{1}{2}\left(\frac{l^n+1}{l+1}\right)\right\}H_m\\&+\frac{s_d\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)\phi q\left(l^n-1\right)}{2(l-1)}+a\log\left(\frac{\phi_0}{\phi}\right)+n\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left(C_{ftm}+C_{fcm}\right)\\&+\frac{q}{\gamma}\left(\frac{l^n-1}{l-1}\right)\left(C_{vtm}+C_{vcm}\right)+\frac{\eta}{2}\rho^2,\end{aligned} \tag{17}$$

where, $0<\phi\le\phi_0$.

4.2.8. Revenue of the manufacturer

Since C_W is the wholesale price and C_M is the unit manufacturing cost, therefore the revenue is

$$\left(C_W-C_M\right)\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right). \tag{18}$$

4.2.9. Profit of the manufacturer

From equations (18) and (17), the manufacturer's profit is

$$ATP_M(S,\rho,q,l,I,\phi,n)=\left(C_W-C_M\right)\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)-\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q(l^n-1)}\left(V_0e^{-\mu I}+I\right)$$

$$-q\left\{\frac{\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}}{P}+\left(1-\frac{\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}}{2P}\right)\left(\frac{l^n-1}{l-1}\right)-\frac{1}{2}\left(\frac{l^n+1}{l+1}\right)\right\}H_m$$
$$-\frac{s_d\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)\phi q\left(l^n-1\right)}{2(l-1)}-a\log\left(\frac{\phi_0}{\phi}\right)-n\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left(C_{ftm}+C_{fcm}\right)$$
$$-\frac{q}{\gamma}\left(\frac{l^n-1}{l-1}\right)\left(C_{vtm}+C_{vcm}\right)-\frac{\eta}{2}\rho^2, \tag{19}$$

where, $0<\phi\le\phi_0$.

4.3. Total profit of the supply chain (SC)

Thus, from equations (19) and (9), the total joint profit per cycle of the SC is $JATP\left(S,\rho,q,l,I,\phi,n,A,\pi_x,k\right)=ATP_M\left(S,\rho,q,s,I,\phi,n\right)+ATP_R\left(S,\rho,q,s,A,\pi_x,k\right)$ which is given as follows:

$$JATP=\left(S-C_M\right)\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)-\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q(l^n-1)}\left(V_0e^{-\mu I}+I\right)$$
$$-q\left\{\frac{\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}}{P}+\left(1-\frac{\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}}{2P}\right)\left(\frac{l^n-1}{l-1}\right)-\frac{1}{2}\left(\frac{l^n+1}{l+1}\right)\right\}H_m-\frac{s_d\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)\phi q\left(l^n-1\right)}{2(l-1)}-a\log\left(\frac{\phi_0}{\phi}\right)$$
$$-n\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left(C_{ftm}+C_{fcm}\right)-\frac{q}{\gamma}\left(\frac{l^n-1}{l-1}\right)\left(C_{vtm}+C_{vcm}\right)-\frac{\eta}{2}\rho^2-\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left(A_0e^{-\kappa A}+A\right)$$
$$-\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left\{C_j\left(t_{r,j-1}-t_s\right)+\sum_{i=1}^{j+1}C_i\left(a_i-b_i\right)\right\}-\frac{\left(\xi_1\rho^{\xi_2}+\xi_3\frac{S_{max}-S}{S-S_{min}}\right)(l-1)}{q\left(l^n-1\right)}\left[\pi_0-\beta_0\pi_x+\frac{\beta_0(\pi_x)^2}{\pi_0}\right]\frac{\sigma}{2}$$
$$\left[\sqrt{t_r+\frac{\alpha Q}{P}}\left(\sqrt{1+k^2}-k\right)+(n-1)\left(\sqrt{t_m+k^2\left(t_r+\frac{\alpha Q}{P}\right)}-k\sqrt{t_r+\frac{\alpha Q}{P}}\right)\right]$$
$$-\left[\frac{q\left(l^n+1\right)}{2(l+1)}+\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)\frac{\sigma}{2}\left\{\sqrt{t_r+\frac{\alpha Q}{P}}\left(\sqrt{1+k^2}-k\right)+(n-1)\left(\sqrt{t_m+k^2\left(t_r+\frac{\alpha Q}{P}\right)}-k\sqrt{t_r+\frac{\alpha Q}{P}}\right)\right\}\right]H_r, \tag{20}$$

where, $0<\phi\le\phi_0$, $0\le\beta_0<1$, $0\le\pi_x\le\pi_0$.

5. The problem and its solution methodology

Since in real scenarios, it is very tough to find the distribution function for the LT demand. The model is thus solved by DFA to find

$$\max JATP\left(S,\rho,q,l,I,\phi,n,A,\pi_x,k\right) \tag{21}$$

subject to $0<\phi\le\phi_0$, $0\le\beta_0<1$, $0\le\pi_x\le\pi_0$.

5.1. Solution methodology

The profit function of the SC is as follows:

$$J\left(S,\rho,q,l,I,\phi,n,A,\pi_x,k\right)=\left(S-C_M-\frac{q}{P}H_m-\frac{1}{\alpha Q}R_1-\frac{\bar{\pi}}{\alpha Q}R_3\right)D-(\alpha Q)R_2+\frac{q}{2}\left(\frac{l^n+1}{l+1}\right)H-(1-\beta)R_3H_r-a\log\frac{\phi_0}{\phi}-\frac{\eta}{2}\rho^2. \tag{22}$$

Equating the partial derivative (B.1), (B.2), (B.5), (B.7), (B.6), (B.4), (B.8), (B.3), and (B.9) in Appendix B to zero, stationary values are obtained as

$$q^*=\frac{qR_7}{\frac{D}{P}H_m-\left(\frac{l^n+1}{l+1}\right)H}R_5,\ l^*=\frac{l^{n+1}-(l+1)^2\frac{L_4}{qH}R_7}{nl^{n-1}}-1,\ S^*=S_{min}+\sqrt{\left(S_{max}-S_{min}\right)\frac{R_5}{D}},\ \rho^*=\left[\frac{\eta}{\xi_1\xi_2R_5}\right]^{\frac{1}{\xi_2-2}},$$

$$\pi_x^*=\frac{1}{2}\left(\pi_0-\frac{1}{2D}\alpha Q\sigma H_r\right),\ k^*=\sqrt{k^2-1}\left[1+(n-1)\left\{\frac{k\sqrt{\frac{\alpha Q}{P}}+t_r}{\sqrt{t_m+k^2\left(\frac{\alpha Q}{P}+t_r\right)}}-1\right\}\right],\ \phi^*=\frac{2a}{\alpha Qs_dD},A^*=\frac{1}{\kappa}\log\left(\kappa A_0\right),I^*=\frac{1}{\mu}\log\left(\mu V_0\right),$$

Above equations give the optimum values of $S^*,\rho^*,q^*,l^*,I^*,\phi{\cdot}A^*,\pi_x^*,k^*$ which are the continuous decision variables of this model. From the expression of the profit function, it is clear that the closed-form solution is impossible as the profit function is non-linear. For that reason, the model is solved numerically, and the results are discussed in Section 6. The global optimality of the profit function is proved through the following theorem. Here, the values of $H,\beta,\bar{\pi},\alpha Q(q,s),D(\rho,S),C(t_r),R_4,R_5,R_7$, and R_8 are given in Appendix A.

Table 2
Data for LT.

Component of LT	Normal duration (days)	Minimal duration (days)	Crashing cost ($/day)
1	30	6	$1.4
2	35	5	$1.2
3	26	9	$3.0

Table 3
Input parameters.

Parameters	value	Parameters	value	Parameters	value
V_0	$1500/batch	μ	0.0012	ξ_1	52
ξ_2	0.68	ξ_3	1	P	100 unit
s_d	$ 30/lot	a	0.0891	ϕ_0	0.00002
C_{ftm}	$0.3/unit	γ	0.5 unit	C_{vtm}	$0.2/container
C_{fcm}	$0.2/unit	C_{vcm}	$0.1/container	A_0	$40/batch
π_0	$150/unit	β_0	0.5	σ	7
H_r	$1.01/unit/unit time	H_m	$0.02/unit/unit time	η	380
S_{max}	$490/unit	S_{min}	$115/unit	t_r	$0.5/unit
t_m	$1.9/unit	C_M	$15/unit	C_W	$65/unit
κ	0.25				

Table 4
Optimal values of dependent variables.

	Parameters	Value
D	demand	87.57 unit
k_1	safety stock of batches other than batch 1	1.15 unit
Q	ordered quantity	110 unit
αQ	delivered quantity	81.64 unit
α	percentage of ordered quantity manufactured by the manufacturer	74.21%
	unreliability	25.79%

Table 5
Optimal results.

	Decision variables	Value
q	first lotsize	26.49 unit/cycle
l	increasing rate of lotsize	1.03
S	selling price	$129.15/unit
ρ	service label	94%
π_x	backorder price discount	$78.23/unit
ϕ	probability of out-of-order stage	0.000001
k	safety stock	1.52 unit
A	OC reduction investment	$14.71/production batch
I	SEC reduction investment	$489.82/production batch
n	number of shipments	3
$JATP$	TP of the SC	$7246.28/cycle

Theorem 1. *The Hessian matrix for the profit function* $JATP\left(q, l, S, \rho, \pi_x, \phi, k, A, I\right)$ *is negative definite at* $\left(q^*, l^*, S^*, \rho^*, \pi_x^*, \phi^*, k^*, A^*, I^*\right)$.

Proof. See Appendix D. □

6. Numerical example

A numerical experiment is illustrated for the judgment of this model below. The input parameters which are taken from Hota et al. (2020) are expressed in Table 3. The MATLAB 12 software is used to get the LT data, the optimal values for the decision variables, which are selling price (S), service level (ρ), lotsize initial value (q), lotsize increasing rate (l), investment for SEC reduction (I), probability of the system goes to out-of-order stage (ϕ), number of shipment (n), investment for OC reduction (A), backorder price discount (π_x), safety stock (k) and the maximum profit ($JATP$) of the SC.

Table 2 gives the LT data, Table 3 gives the values of the parameters taken in this study, the dependent variable's optimum outcomes and the amount of order are in Table 4 gives, and Table 5 gives the maximum profit and the decision variable's optimum values.

The dependent variables of this study are demand (D), the amount delivered to the retailer by the manufacturer (αQ), and safety stocks of the batches other than batch 1 (k_1). The demand D depends on selling price (S) and service label (ρ); the delivered amount (αQ) depends on lot size initially (q), lot size increasing rate (l) and the number of shipments (n); safety stocks of the batches other than batch 1 (k_1) depend on the safety stock of batch 1 (k). From the amount of ordered quantity and amount of delivered product, the unreliability is evaluated.

6.1. Numerical proof of global optimality

At the optimal point $\left(q^*, l^*, S^*, \rho^*, \pi_x^*, \phi^*, k^*, A^*, I^*\right)$ the Hessian matrix of $JATP$ is as follows:

$$H = \begin{pmatrix} -3.87 & -52.4 & -1.3 & 24.7 & 0.0005 & -3494.79 & -2.1 & 0 & 0 \\ -52.44 & -1814.4 & -33.7 & 648.3 & 0.01 & -91651.4 & -55.1 & 0 & 0 \\ -1.28 & -33.7 & -2.6 & 36.07 & 0.0006 & 2303.51 & -1.9 & 0 & 0 \\ 24.71 & 648.3 & 36.06 & -791.25 & -0.01 & -44296.5 & 0 & 0 & 0 \\ 0.0004 & 0.01 & 0.0006 & -0.01 & -0.007 & 0 & 0 & 0 & 0 \\ -3494.79 & -91651.4 & 2303.51 & -44296.5 & 0 & -8.9\times 10^{10} & 0 & 0 & 0 \\ -2.10 & -55.1 & -1.9 & 38.3 & 0.02 & 0 & -78.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.06 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.001 \end{pmatrix}$$

The eigen values of the Hessian matrix are -8.91×10^{10}, -2132.81, -476.951, -77.9359, -2.29691, -0.890257, -0.0582, -0.00739989, -0.0011, which are all negative. Also the principal minors are $H_{11} = -3.8797 < 0$, $H_{22} = 4288.67 > 0$, $H_{33} = -8096.29 < 0$, $H_{44} = 2.09302 \times 10^6 > 0$, $H_{55} = -15488 < 0$, $H_{66} = 1.37971 \times 10^{15} > 0$, $H_{77} = -1.06887 \times 10^{17} < 0$, $H_{88} = 6.22084 \times 10^{15} > 0$, $H_{99} = -6.84293 \times 10^{12} < 0$ which are alternates in sign started from negative sign. Both results show that the profit function is maximum at this optimal point.

6.2. Special cases

6.2.1. Case 1. Without SEC reduction investment

For reducing the SEC, a discrete type of investment is applied in this study where the investment I is a decision variable. But there is no such investments in the traditional SC models. For examining the advantage of the variable interment for SEC reduction, instead of any investment, a fixed setup cost $V_0 = \$2680$ is applied and the profit $JATP$ is obtained as $JATP = \$298.62$. The values of the other decision variables become $S = \$129.23$/unit, $\rho = 0.94\%$, $q = 6.43$ unit/cycle, $l = 1.01$ unit, $\phi = 0.000001$,n= 3,A= 14.71\$/production batch, $\pi_x = \$78.22$/unit, $k = 1.52$ units.

There is a huge loss (95.87%) in profit. Thus, to avoid uncertainty regarding profit, the investment for the SEC reduction is essential and has to be considered as a variable.

6.2.2. Case 2. Without OC reduction investment

To reduce OC, another discrete variable investment, (A), is applied in this study. Now the question is, is it profitable or not? To find the answer, instead of investing for OC reduction, a fixed OC of \$15/batch is applied in the final numerical file of the study, and the profit $JATP$ is obtained as $JATP = \$6183.18$/cycle (the remaining data is unchanged). The optimal values of the other decision variables become $S = \$129.19$/unit, $\rho = 0.94\%$, $q = 17$ unit/cycle, $l = 1.02$ unit, $I = 489.82\$$/production batch, $\phi = 0.000001, n = 3, \pi_x = \$78.22$/units, $k = 1.52$ units.

Thus, although the OC is lower, the SC loses 14.67% of its profit. Therefore, the discrete variable investment for the OC reduction is vital for maximizing the SC's profit.

6.2.3. Case 3. SSSD policy

An SSMUID policy with a variable initial lotsize and an increasing variable rate is applied to save the retailer's holding cost. In the traditional SC model, the SSSD policy was applied, in which the whole produced quantity was delivered in a single lot. Instead of the SSMUID policy, if this SSSD policy is applied in the proposed study, the number of shipments n must be 1, and the increasing rate l must be 0. Now applying $n = 1$ and $l = 0$ and keeping the remaining data the same as previous, the profit $JATP$ becomes $JATP = \$6628$/cycle. The optimum outcomes are $S = \$122.2$/unit, $\rho = 0.71\%$, $I = \$489.82$/production batch, $\phi = 0.000002$, $A = \$14.71$/production batch, $\pi_x = \$76.55$/units, $k = 0$ units. In this result, we get production quantity (αQ) = the initial lot size (q) (which is the only lot for this case) = 2425.94 unit/cycle. Another notable point is that the production rate (P) = 100/year, which is near equals to the demand (D) = 101.01/year, and hence the safety stock (k) becomes 0.

The profit using the SSSD policy is less than the profit of the SC when the manufacturer applies the SSMUID policy. Thus, the industry can benefit more from the SSMUID policy than the SSSD policy.

6.2.4. Case 4. SSMD policy

Another popular shipment strategy used in SC models is the SSMD policy. In this policy, the manufacturer produces the whole amount at a time but delivered them to the retailer in multiple lots with equal lot sizes. If the manufacturer applies SSMD policy in this study, the whole produced quantity is delivered in n lots with equal size, and then the increased rate l must be 1. Thus applying $l = 1$ in the final numerical values and with the remaining data unchanged. The decision variable's optimum outcomes and JATP become, $S = \$129.07$/unit, $\rho = 0.94\%$, $q = 28.77$ unit/cycle, $I = \$489.82$/production batch, $\phi = 0.000001$,n= 3,A= \$14.71/production batch, $\pi_x = \$183.4$/units, $k = 1.53$ units and $JATP = \$7163.65$/cycle. The total produced volume (αQ) = 86.31 units/cycle which are delivered to the retailer in $n = 3$ lots with equal lot size $q = 28.77$ unit/cycle.

Thus, industries get more profit if they apply SSMUID policy than SSMD policy. To get maximum profit, the backorder price discount given by the retailer is $\pi_x = \$183.4$/unit which is very high than SSMUID policy.

6.2.5. Case 5. No backorder price discount

LT may increase during the shortage period due to the unreliable manufacturer. But customers want it quick and on time delivery. For that reason, the retailer may loose customers. In the proposed study, the retailer gives some discount to attract customers for waiting up to replenishment, which called backorder price discount. The number of backorder depends on this discount. But there are so many emergency situations in which the retailer is unable to give any backorder price discount. To examine this situation applying $\pi_x = \$0$/unit in the final numerical file and keeping the remaining data unchanged, the result is obtained as $S = \$129.15$/unit, $\rho = 0.94\%$, $q = 26.58$ unit/cycle, $l = 1.02$ units, $I = \$489.82$/production batch,

Table 6
Sensitivity analysis.

Parameters	% changes	S	ρ	q	l	I	A	π_x	$JATP$ (%)
V_0	−50%	+1.09	+4.8	NF	NF	NF	–	–	+35.27
	−25%	+0.55	+2.5	NF	NF	−239.74	–	–	+36.30
	+25%	−0.58	−3.8	+150.8	+0.15	+185.95	–	–	+18.75
	+50%	−1.19	−5	+301.26	+0.30	+337.88	–	–	+14.91
A_0	−50%	+0.03	+0.2	−7.59	−0.02	–	−10.03	–	−10.56
	−25%	+0.01	+0.1	−3.79	−0.01	–	−4.17	–	−4.39
	+25%	−0.01	−0.1	+3.79	+0.01	–	+3.23	–	+3.27
	+50%	−0.03	−0.2	+7.59	+0.02	–	+5.87	–	+5.81
H_m	−50%	–	–	+3.18	–	–	–	–	+2.82
	−25%	–	–	+1.59	–	–	–	–	+1.49
	+25%	–	–	−1.59	–	–	–	–	−1.69
	+50%	–	–	−6.38	–	–	–	–	−3.63
H_r	−50%	–	–	+30.27	+0.03	–	–	−39.1	+13.59
	−25%	–	–	+10.07	+0.01	–	–	−19.65	+7.20
	+25%	–	–	−6.03	−0.01	–	–	+19.65	−8.02
	+50%	–	–	−10.05	−0.03	–	–	+39.1	−17.01
s_d	−50%	+0.04	+0.04	+12.9	+0.01	–	–	–	+8.37
	−25%	+0.02	+0.03	+6.45	–	–	–	–	+5.06
	+25%	−0.02	–	−6.45	–	–	–	–	−8.52
	+50%	−0.04	−0.02	−12.9	−0.01	–	–	–	−25.50
C_M	−50%	+2.19	NF	–	–	–	–	–	+4.48
	−25%	+1.14	+5	–	–	–	–	–	+2.12
	+25%	−1.23	−5	–	–	–	–	–	−1.73
	+50%	−2.6	−9	–	–	–	–	0	−2.82

NF - Not feasible, – No change.

$\phi = 0.000001$,n= 3,A= \$14.71/ production batch, $k = 1.52$ units and $JATP = \$7166.02$/cycle and it is seen that the SC get less profit in this case. Thus, the variable backorder price discount is appropriate for this proposed study to get maximum profit.

7. Sensitivity analysis

The change of $JATP$ that occurs due to the change of cost parameters is shown here and analyzes its importance. The changes in values of decision variables selling price (S), service level (ρ), first lotsize (q), increasing rate (l), investment for SEC reduction (I), investment for OC reduction (A), backorder price discount (π_x), and $JATP$ for reduction and increase of cost parameters initial SEC (V_0), initial OC (A_0), HC of the retailer (H_r) and manufacturer (H_m), defective cost (s_d), and the manufacturing cost (C_M) by 25% and 50% are described in Table 6 which suggests the industry manager to handle those parameters carefully. The observations from Table 6, are mentioned below:

- In general, the profit increases by decreasing the cost. From Table 6, it is found that a decrease of the initial SEC (V_0) increases the total joint profit $JATP$ of the SC and, on the other hand, increase of V_0 decreases $JATP$, which is quite natural. But if the changes of decision variables are noticed, one can see no changes in A and π_x, but S and ρ increase and q, l, I diverge for the change of V_0. Thus, V_0 is a very sensitive parameter for this model, and the industry manager is advised to be careful about this parameter.
- An increase of the initial OC (A_0) by 25% and 50% increase the decision variables q, l, A and the profit $JATP$ but decrease the decision variables S and ρ. On the other hand, decrease of the initial OC (A_0) by 25% and 50%, $1, l, A$, and $JATP$ decreases but S, ρ increase. The decision variables A and π_x have no effect on the OC change.
- If the HC of the manufacturer (H_m) is increased or decreased by 50% and 25% then the initial lotsize q and profit $JATP$ are decreased and increased, respectively. On the other hand, the change in H_m cannot change the decision variables. Thus, the industry manager should be careful about the place of the manufacturing house. The holding cost of the place must be cheap to increase profit. It is the least sensitive parameter of this study.
- The HC of the retailer H_r is the most sensitive parameter of the model. By decreasing H_r, there is a huge growth in initial lotsize q and the profit $JATP$, and on the other hand, with an increase in H_m, those values hugely decrease. The increasing rate of lot size l has a minor increase and minor decrease according to the increase and decrease of H_r. The decision variables S, ρ, I, π_x remain unchanged for the change of H_r.
- If the value of defective cost s_d is increased by 25% and 50%, then the decision variables S, ρ, q, l decrease and the decision variables I, A, π_x remain unchanged. The profit $JATP$ hugely decreases. Again, if s_d is decreased by 25% and 50%, decision variables S, ρ, q, l increase; I, A, π_x remain unchanged and the profit $JATP$ hugely increases. Thus, it is important to control the defective cost to maximize profit.
- The result of an increase or decrease in C_M has an impact on S, ρ and the total profit. Increasing C_M by 25% and 50%, it is found that the selling price and total profit decrease, and they are increased when C_M is decreased by 25% and 50%. But the service level becomes divergent when C_M is decreased by 50%.

The change in total profit based on the change of various parameters from −50% to +50% is represented graphically in Fig. 5. From Fig. 5, it may discuss that:

- Based on changes on the initial SEC V_0, the graph of the profit $JATP$ is initially going upward slightly, and then it goes downward. But ultimately, it is always above the X axis, which implies that the profit $JATP$ never decreases; only its increased amount varies.

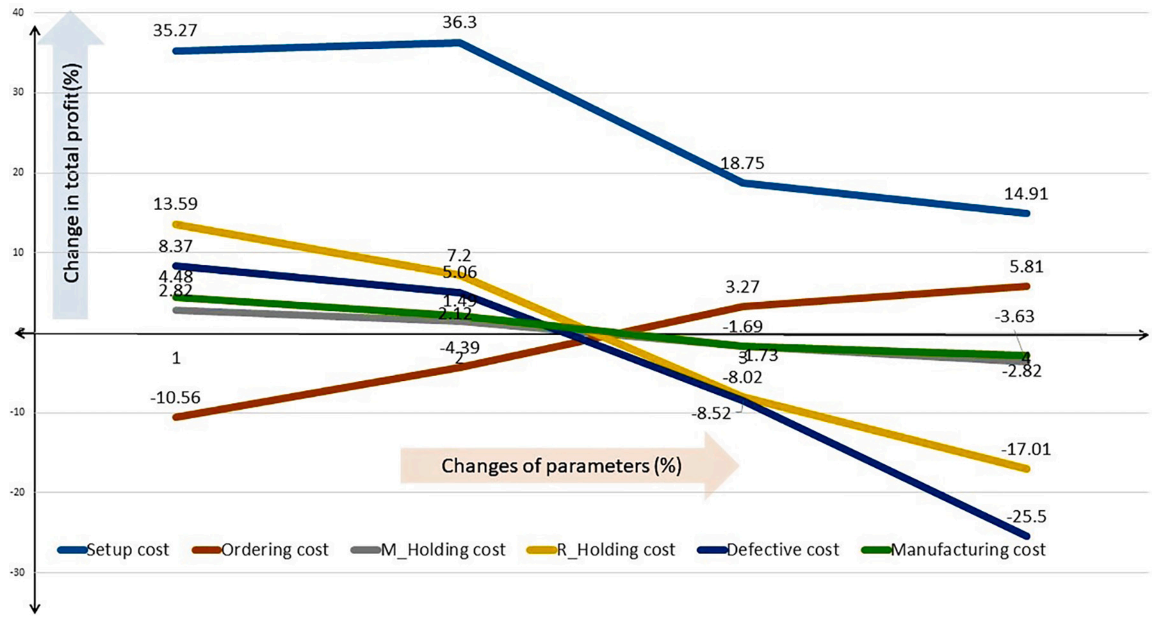

Fig. 5. Sensitivity analysis graph.

- Based on the changes of the parameters H_r, s_d, C_M and H_m, the profit graph goes down that means the point $JATP$ decreases for increasing the values of the said parameters.
- Based on the changes on the initial ordering cost A_0, the graph of the profit $JATP$ is initially going upward, which confirms that the profit $JATP$ increases for increasing value of A_0.

8. Managerial insights

The insights from this study are as follows:

- Quick and on time delivery is a very serious factor for a product to increase its reputation. However, in today's environment, unreliability of player's increases LT, consequently shortages arise, and reputation decreases. Here, the unreliable manufacturer delivers less than the ordered amount to the retailer. The manufacturer hides the information regarding the delivery time, which increases the LT. Unreliability in both of these areas creates a shortage, which decreases the reputation of the retailer. Solving this unreliability issue, therefore, represents a big challenge for the industry, and this study presents a way to solve this problem.
- Usually, the time duration between proposed order and delivery of the products known as LT. However, it is formed of many parts. A reduction of any one of these parts may reduce the LT. In this study, processing time of raw material has been reduced as well as the variable backorder rate.
- When shortages arise, customers may wait for the next delivery or go to other retailers to fulfill their demand. To reduce the number of lost sales, the retailer offers a backorder price discount to attract customers. In traditional SCs, the price discount is considered as a constant, but it may vary. In this study, the total profit is optimized by considering the backorder price discount as a variable and through SSMUD policy.
- Maximum SC models consider SSSD policy, while others consider SSMD. However, the demand for a product is not always the same, and the HC of the retailer is usually higher than the HC of the manufacturer. Therefore, it is unscientific for an industry to send all of the product in a single lot or to send all the product in equal lots. This study maximizes the TP through increasing demand with a two-stage safety stock.
- In terms of reducing shortages, due to the unreliability of the manufacturer, in this study, two-stage safety stock is applied. Thus, by using this policy, industry managers can solve unreliability problem and maximize the TP of the SC.

9. Conclusions and future extensions

An SC with an unreliable manufacturer, SSMUD policy, investments for SEC, OC, quality and service level with price, and service-dependent demand were discussed in the study. Because of the unreliability of the manufacture, the retailer faced shortage. To solve the shortage problem, a two-stage safety stock with a variable backorder rate and a variable backorder price discount was applied. That backorder price discount saved the retailer from more lost sales due the increased LT, because of unreliable manufacturer. As the LT demand was random and the information of the distribution function was not known, still the retailer could manage the profit by maintaining the collaboration of the SC. Finally, the total SC profit was maximized using DFA. In comparison, in a traditional SC, the players were usually reliable. In this study, the profit of the SC was maximized in several cases and the proof was done to show the best strategy for the retailer to save from excessive lost sales. Considering SSMUID policy with a variable first lot size and variable increasing rate, the total profit of the SC was maximized. Several special cases were explained, and from these special cases, it could be concluded that the SSMUID technique was mostly profitable than the SSSD or SSMD technique when the retailer can control the amount of the lost sales. Furthermore, variable investment for the SEC reduction and OC reduction gave more profit than fixed values. It was established that the variable probability of the *out-of-order* stage gave better results than fixed value. An algorithm was developed to discuss the methodology.

The study can be extended to multi-echelon SCM (Nasiri et al., 2021). A discrete investment for SEC reduction is one of the limitations of the study. One can use continuous investment (Sepehri et al., 2021) to reduce the SEC. This model can be extended by applying smart manufacturing (Saha et al., 2023) and dynamic investments (Singh et al., 2023). Offering some discount policies and solving it in decentralized case by Stackelberg game (Ali et al., 2018) may be another extension of this study. This model can be extended by RFID (Sarkar and Guchhait, 2023) and technological growth (Hota et al., 2022) to prevent the unreliability. The study can be extended by applying advanced inspection policies (Sarkar et al., 2022a), which is one of the major problems in supply chain.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data has been provided in the manuscript.

Appendix A

The values are as follows:

$$H = H_m - H_r, \quad \beta = \frac{\beta_0 \pi_x}{\pi_0}, \quad \bar{\pi} = \pi_0 - \beta_0 \pi_x + \frac{\beta_0 (\pi_x)^2}{\pi_0}, \quad \alpha Q(q,s) = q\left(\frac{l^n - 1}{l - 1}\right)$$

$$D(\rho, S) = \xi_1 \rho^{\xi_2} + \xi_3 \left(\frac{S_{max} - S}{S - S_{min}}\right), \quad C(t_r) = \left[C_j\left(t_{r,j-1} - t_s\right)) + \sum_{i=1}^{j+1} C_i (a_i - b_i)\right].$$

$F_1 = n\left(C_{ftm} + C_{fcm}\right) > 0, F_2 = \frac{1}{\gamma}\left(C_{vtm} + C_{vcm}\right), H = \frac{1}{2}\left(H_m - H_r\right) < 0, S_1 = \frac{S_{max}-S}{S-S_{min}} > 0, S_2 = -\frac{S_{max}-S}{\left(S-S_{min}\right)^2} - \frac{1}{S-S_{min}} = -\frac{S_{max}-S_{min}}{(S-S_{min})^2} < 0, S_3 = \frac{S_{max}-S}{\left(S-S_{min}\right)^3} + \frac{1}{\left(S-S_{min}\right)^2} > 0, L_1 = \frac{nl^{n-1}}{l+1} - \frac{l^n+1}{(l+1)^2} > 0, L_2 = \frac{(l-1)nl^{n-1}}{(l^n-1)^2} - \frac{1}{l^n-1} = \frac{(n-1)l^n - \left(nl^{n-1}-1\right)}{(l^n-1)^2}, L_3 = \frac{nl^{n-1}}{l-1} - \frac{l^n-1}{(l-1)^2} = \frac{(n-1)l^n - \left(nl^{n-1}-1\right)}{(l-1)^2}, L_4 = \frac{(n-1)l^n - \left(nl^{n-1}-1\right)}{(l-1)(l^n-1)}, L_2\left(\frac{\alpha Q}{q}\right) = L_4, L_3\left(\frac{q}{\alpha Q}\right) = L_4, L_5 = \frac{(n-1)nl^{n-2}}{l+1} - \frac{2nl^{n-1}}{(l+1)^2} + \frac{2(l^n+1)}{(l+1)^3}, L_6 = -\frac{2(l-1)n^2 l^{2n-2}}{(l^n-1)^3} + \frac{(l-1)(n-1)nl^{n-2}}{(l^n-1)^2} + \frac{2nl^{n-1}}{(l^n-1)^2}, L_7 = \frac{(n-1)nl^{n-2}}{l-1} - \frac{2nl^{n-1}}{(l-1)^2} + \frac{2(l^n-1)}{(l-1)^3}, R_1 = S - C_M - \frac{qH_m}{P} > 0,$

$R_2 = \frac{1}{\alpha Q}\left(V_0 e^{-muI} + I + A_0 + e^{\kappa A} + C(t_r) + F_1\right) > 0, R_3 = \alpha Q\left(\frac{H_m}{2P} - \frac{1}{2}\phi s_d\right),$

$R_4 = \frac{\sigma}{2\alpha Q}\left[\left(\sqrt{k^2+1} - k\right)\sqrt{\frac{\alpha Q}{P} + t_r}, +(n-1)\left\{\sqrt{k^2\left(\frac{\alpha Q}{P} + t_r\right) + t_m} - k\sqrt{\frac{\alpha Q}{P} + t_r}\right\}\right] > 0, R_5 = R_1 - R_2 + R_3 - \bar{\pi} R_4,$

$R_6 = \frac{\sigma}{4P}\left[\frac{\sqrt{k^2+1}-k}{\sqrt{\frac{\alpha Q}{P}+t_r}} + (n-1)\left\{\frac{k^2}{\sqrt{k^2\left(\frac{\alpha Q}{P}+t_r\right)+t_m}} - \frac{k}{\sqrt{\frac{\alpha Q}{P}+t_r}}\right\}\right]$

$R_7 = \frac{1}{q}\left[D\left(R_2 + R_3 + \bar{\pi} R_4\right) - \alpha Q\left(F_2 + H_m\right) + R_6\left\{\frac{1}{2}\alpha Q \sigma H_r\left(1 - \frac{\beta_0 \pi_x}{\pi_0}\right) - \bar{\pi} D\right\}\right]$

$R_8 = \frac{\sigma}{2\alpha Q}\left[\left(\frac{k}{\sqrt{k^2+1}} - 1\right)\sqrt{\frac{\alpha Q}{P} + t_r} + (n-1)\left\{\frac{k\left(\frac{\alpha Q}{P}+t_r\right)}{\sqrt{k^2\left(\frac{\alpha Q}{P}+t_r\right)+t_m}} - \sqrt{\frac{\alpha Q}{P} + t_r}\right\}\right]$

$R_9 = \sigma\left(\frac{\alpha Q}{4P}\right)^2\left[-\frac{\sqrt{k^2+1}-k}{\left(\frac{\alpha Q}{P}+t_r\right)^{3/2}} + (n-1)\left\{\frac{k}{\left(\frac{\alpha Q}{P}+t_r\right)^{3/2}} - \frac{k^4}{\left(k^2\left(\frac{\alpha Q}{P}+t_r\right)+t_m\right)^{3/2}}\right\}\right]$

$R_{10} = \frac{\sigma}{4P}\left[\frac{\frac{k}{\sqrt{k^2+1}}-1}{\sqrt{\frac{\alpha Q}{P}+t_r}} + (n-1)\left\{\frac{2k}{\sqrt{k^2\left(\frac{\alpha Q}{P}+t_r\right)+t_m}} - \frac{k^3\left(\frac{\alpha Q}{P}+t_r\right)}{\left(k^2\left(\frac{\alpha Q}{P}+t_r\right)+t_m\right)^{3/2}} - \frac{1}{\sqrt{\frac{\alpha Q}{P}+t_r}}\right\}\right]$

$R_{11} = \frac{\sigma}{2}\left[\frac{\sqrt{\frac{\alpha Q}{P}+t_r}}{\left(k^2+1\right)^{3/2}} + (n-1)\left\{\frac{\frac{\alpha Q}{P}+t_r}{\sqrt{k^2\left(\frac{\alpha Q}{P}+t_r\right)+t_m}} - \frac{k^2\left(\frac{\alpha Q}{P}+t_r\right)^2}{\left(k^2\left(\frac{\alpha Q}{P}+t_r\right)+t_m\right)^{3/2}}\right\}\right]$

$R_{12} = R_2 + R_3 + \bar{\pi}\left(R_4 - R_6\right), R_{13} = \frac{L_4}{q}\left[-D\left\{R_2 - R_3 + \bar{\pi}\left(R_4 - 2R_6\right)\right\} - \alpha Q\left(F_2 + H_m\right) + \bar{\bar{\pi}}\left(\frac{2R_9}{\alpha Q} + R_6\right)\right],$

$R_{14} = L_6 \frac{\alpha Q}{q} D\left(R_2 + \bar{\pi} R_4\right) + L_7 \frac{q}{\alpha Q}\left\{DR_3 + \bar{\bar{\pi}} R_6 - \alpha Q\left(F_2 + H_m\right)\right\} + 2\left(L_4\right)^2\left(\bar{\pi} D R_6 + \frac{\bar{\bar{\pi}}}{\alpha Q} R_9\right),$

$R_{15} = -\frac{H_m}{P} + \frac{1}{q} R_{12}, \bar{\pi} = \pi_0 - \beta_0 \pi_x + \frac{\beta_0 \pi_x^2}{\pi_0}, \Pi_1 = \frac{1}{2}\alpha Q \sigma H_r\left(1 - \frac{\beta_0 \pi_x}{\pi_0}\right) - \bar{\pi} D, \Pi_2 = \frac{2\pi_x}{\pi_0} - 1.$

Appendix B. Partial derivatives

The partial differentiations of $JATP$ function (equation (21)) with respect to decision variables are as follows (J_x implies $\frac{\partial J}{\partial x}$):

$$J_q = \left(\frac{l^n + 1}{l + 1}\right) H - \frac{D}{P} H_m + R_7, \tag{B.1}$$

$$J_l = q\left(L_1 H + L_4 R_7\right), \tag{B.2}$$

$$J_S = D + S_2 R_5, \tag{B.3}$$

$$J_\rho = -\eta\rho + \xi_1\xi_2\rho^{\xi_2-1}R_5, \quad (B.4)$$

$$J_{\pi_x} = -\left\{ D\Pi_2 + \alpha Q\frac{\sigma}{2\pi_0}H_r \right\}\beta_0 R_4, \quad (B.5)$$

$$J_\phi = -\frac{1}{2}\alpha Q D s_d + \frac{a}{\phi}, \quad (B.6)$$

$$J_k = \bar{\bar{\pi}} R_8, \quad (B.7)$$

$$J_A = -\frac{D}{\alpha Q}\left(1-\kappa A_0 e^{-\kappa A}\right), \quad (B.8)$$

$$J_I = -\frac{D}{\alpha Q}\left(1-\mu V_0 e^{-\mu I}\right). \quad (B.9)$$

Appendix C. Hessian matrix

The structure of the Hessian matrix (when n given) is as follows

$$H = \begin{bmatrix} J_{qq} & J_{ql} & J_{qS} & J_{q\rho} & J_{q\pi_x} & J_{q\phi} & J_{qk} & J_{qA} & J_{qI} \\ J_{lq} & J_{ll} & J_{lS} & J_{l\rho} & J_{l\pi_x} & J_{l\phi} & J_{lk} & J_{lA} & J_{lI} \\ J_{Sq} & J_{Sl} & J_{SS} & J_{S\rho} & J_{S\pi_x} & J_{S\phi} & J_{Sk} & J_{SA} & J_{SI} \\ J_{\rho q} & J_{\rho l} & J_{\rho S} & J_{\rho\rho} & J_{\rho\pi_x} & J_{\rho\phi} & J_{\rho k} & J_{\rho A} & J_{\rho I} \\ J_{\pi_x q} & J_{\pi_x l} & J_{\pi_x S} & J_{\pi_x\rho} & J_{\pi_x\pi_x} & J_{\pi_x\phi} & J_{\pi_x k} & J_{\pi_x A} & J_{\pi_x I} \\ J_{\phi q} & J_{\phi l} & J_{\phi S} & J_{\phi\rho} & J_{\phi\pi_x} & J_{\phi\phi} & J_{\phi k} & J_{\phi A} & J_{\phi I} \\ J_{kq} & J_{kl} & J_{kS} & J_{k\rho} & J_{k\pi_x} & J_{k\phi} & J_{kk} & J_{kA} & J_{kI} \\ J_{Aq} & J_{Al} & J_{AS} & J_{A\rho} & J_{A\pi_x} & J_{A\phi} & J_{Ak} & J_{AA} & J_{AI} \\ J_{Iq} & J_{Il} & J_{IS} & J_{I\rho} & J_{I\pi_x} & J_{I\phi} & J_{Ik} & J_{IA} & J_{II} \end{bmatrix}$$

Where $J = JATC\left(q,l,S,\rho,\pi_x,\phi,k,A,I\right)$ and $J_{xy} = \frac{\partial^2 J}{\partial x \partial y}$.

The following expressions indicate the partial derivatives of second order at $\left(q^*, l^*, S^*, \rho^*, \pi_x^*, \phi^*, k^*, A^*, I^*\right)$.

$J_{qq} = \frac{2}{q^2}\left\{\frac{\Pi_1}{\alpha Q}R_9 - D\left(R_{12}-R_3\right)\right\}, J_{lq} = HL_1 + R_{13},$
$J_{Sq} = S_2R_{15}, J_{Sq} = S_2R_{15}, J_{\rho q} = \xi_1\xi_2\rho^{\xi_2-1}R_{15},$
$J_{\pi_x q} = \left[\Pi_2 D R_4 - \left\{D\Pi_2 + \alpha Q\frac{\sigma}{2\pi_0}H_r\right\}R_6\right]\frac{\beta_0}{q} = \frac{1}{q}\Pi_2 D\beta_0 R_4,$
$J_{\phi q} = -\frac{1}{2q}D\alpha Q s_d = -\frac{1}{q}\frac{ab}{\phi}, J_{kq} = \frac{1}{q}\left(\bar{\pi}DR_8 + \Pi_1 R_{10}\right) = \frac{1}{q}\Pi_1 R_{10},$
$J_{Aq} = 0, J_{Iq} = 0.$
$J_{ql} = HL_1 + R_{13}, J_{ll} = L_5 qH + R_{14}, J_{Sl} = L_4 S_2 R_{12},$
$J_{\rho l} = L_4\xi_1\xi_2\rho^{\xi_2-1}R_{12}, J_{\pi_x l} = L_4\Pi_2 DR_4\beta_0, J_{\phi l} = -L_4\frac{ab}{\phi},$
$J_{kl} = L_4\left(\bar{\pi}DR_8 + \Pi_1 R_{10}\right) = L_4\Pi_1 R_{10}, J_{Al} = 0, J_{Il} = 0.$
$J_{qS} = S_2R_{15}, J_{lS} = S_2L_4R_{12}, J_{SS} = 2\left(S_2 + S_3R_5\right), J_{\rho S} = \xi_1\xi_2\rho^{\xi_2-1},$
$J_{\pi_x S} = -S_2\Pi_2\beta_0R_4, J_{\phi S} = -\frac{1}{2}\alpha Q S_2 s_d = -\frac{S_2}{D}\frac{ab}{\phi}, J_{kS} = -S_2\bar{\pi}R_4,$
$J_{AS} = 0, J_{IS} = -\frac{1}{\alpha Q}S_2\left(1-\mu V_0 e^{-\mu I}\right) = 0.$
$J_{q\rho} = \xi_1\xi_2\rho^{\xi_2-1}R_{15}, J_{l\rho} = \xi_1\xi_2\rho^{\xi_2-1}R_{12}L_4, J_{S\rho} = \xi_1\xi_2\rho^{\xi_2-1},$
$J_{\rho\rho} = -\eta + \xi_1\xi_2\left(\xi_2-1\right)\rho^{\xi_2-2}R_5 = \frac{\eta}{2}\left(\xi_2-2\right), J_{\pi_x\rho} = -\xi_1\xi_2\rho^{\xi_2-1}\Pi_2\beta_0R_4,$
$J_{\phi\rho} = -\frac{1}{2}\xi_1\xi_2\rho^{\xi_2-1}\alpha Q s_d = -\frac{\xi_1\xi_2\rho^{\xi_2-1}}{D}\frac{ab}{\phi}, J_{k\rho} = -\xi_1\xi_2\rho^{\xi_2-1}\bar{\pi}R_8 = 0,$
$J_{A\rho} = \frac{\xi_1\xi_2\rho^{\xi_2-1}}{\alpha Q}\left(\kappa A_0 e^{-\kappa A} - 1\right) = 0, J_{I\rho} = \frac{\xi_1\xi_2\rho^{\xi_2-1}}{\alpha Q}\left(\mu V_0 e^{-\mu I} - 1\right) = 0.$
$J_{q\pi_x} = \left[\Pi_2 DR_4 - \left\{D\Pi_2 + \alpha Q\frac{\sigma}{2\pi_0}H_r\right\}R_6\right]\frac{\beta_0}{q} = \frac{1}{q}\Pi_2 D\beta_0 R_4,$
$J_{l\pi_x} = \left[\Pi_2 DR_4 - \left\{D\Pi_2 + \alpha Q\frac{\sigma}{2\pi_0}H_r\right\}R_6\right]\beta_0 L_4 = L_4\Pi_2 DR_4\beta_0,$
$J_{S\pi_x} = -S_2\Pi_2\beta_0R_4, J_{\rho\pi_x} = -\xi_1\xi_2\rho^{\xi_2-1}\Pi_2\beta_0R_4, J_{\pi_x\pi_x} = -\frac{2}{\pi_0}D\beta_0R_4,$
$J_{\phi\pi_x} = 0, J_{k\pi_x} = -\left\{D\Pi_2 + \alpha Q\frac{\sigma}{2\pi_0}H_r\right\}\beta_0R_8 = 0, J_{A\pi_x} = 0, J_{I\pi_x} = 0.$
$J_{q\phi} = -\frac{1}{q}\frac{ab}{\phi}, J_{l\phi} = -L_3 q\frac{1}{2}Ds_d = -L_4\alpha Q\frac{1}{2}Ds_d = -L_4\frac{ab}{\phi},$
$J_{S\phi} = -S_2\frac{1}{2}\alpha Q s_d = -\frac{S_2}{D}\frac{ab}{\phi}, J_{\rho\phi} = -\frac{\xi_1\xi_2\rho^{\xi_2-1}}{D}\frac{ab}{\phi}, J_{\pi_x\phi} = 0,$
$J_{\phi\phi} = -\frac{a}{\phi^2}, J_{k\phi} = 0, J_{A\phi} = 0, J_{I\phi} = 0. J_{qk} = \frac{1}{q}\left(\bar{\pi}DR_8 + \Pi_1R_{10}\right) = \frac{1}{q}\Pi_1R_{10},$
$J_{lk} = L_4\left(\bar{\pi}DR_8 + \Pi_1R_{10}\right) = L_4\Pi_1R_{10}, J_{Sk} = -S_2\bar{\pi}R_4,$
$J_{\rho k} = -\xi_1\xi_2\rho^{\xi_2-1}\bar{\pi}R_8 = 0, J_{\pi_x k} = -\left\{D\Pi_2 + \alpha Q\frac{\sigma}{2\pi_0}H_r\right\}\beta_0R_8 = 0,$
$J_{\phi k} = 0, J_{kk} = \frac{\Pi_1}{\alpha Q}R_{11}, J_{Ak} = 0, J_{Ik} = 0.$
$J_{qA} = \frac{1}{q}\frac{D}{\alpha Q}\left(1-\kappa A_0e^{-\mu A}\right) = 0, J_{lA} = L_2\frac{D}{q}\left(1-\kappa A_0e^{-\mu A}\right) = 0,$
$J_{SA} = -\frac{1}{\alpha Q}S_2\left(1-\kappa A_0e^{-\mu A}\right) = 0 J_{\rho A} = \frac{1}{\alpha Q}\xi_1\xi_2\rho^{\xi_2-1}\left(1-\kappa A_0e^{-\mu A}\right) = 0,$
$J_{\pi_x A} = 0, J_{\phi A} = 0, J_{kA} = 0, J_{AA} = -\frac{D}{\alpha Q}\kappa, J_{IA} = 0.$
$J_{qI} = \frac{1}{q}\frac{D}{\alpha Q}\left(1-\mu V_0e^{-\mu I}\right) = 0, J_{lI} = L_2\frac{D}{q}\left(1-\mu V_0e^{-\mu I}\right) = 0,$

$J_{SI} = -\frac{1}{\alpha Q} S_2 \left(1 - \mu V_0 e^{-\mu I}\right) = 0 J_{\pi_x I} = 0, J_{\rho I} = \frac{1}{\alpha Q} \xi_1 \xi_2 \rho^{\xi_2 - 1} \left(1 - \mu V_0 e^{-\mu I}\right) = 0,$
$J_{\phi I} = 0, J_{kI} = 0, J_{AI} = 0, J_{II} = -\frac{D}{\alpha Q} \mu.$

Appendix D. Principal minors

At the stationary point, the 1st order principal minor is as follows:

$$| H_{11} | = -\frac{2}{q^2} \left[DR_{12} - \left(\frac{\Pi_1}{\alpha Q} R_9 + R_3 \right) \right].$$

The 1st order principal minor is negative if $DR_{12} > \frac{\Pi_1}{\alpha Q} R_9 + R_3$. At the stationary point, the 2nd order principal minor is as follows:

$$| H_{22} | = \begin{vmatrix} J_{qq} & J_{ql} \\ J_{lq} & J_{ll} \end{vmatrix} = \frac{2}{q^2} \Omega_1 - \Omega_2$$

where $\Omega_1 = \frac{\Pi_1}{\alpha Q} qHL_5 R_9 + \frac{\Pi_1}{\alpha Q} R_9 R_{14} + qHL_5 R_3 + R_3 R_{14}$
$\Omega_2 = 2HL_1 R_{13} + \frac{2}{q} DHL_5 R_{12} + \frac{2}{q^2} DR_{12} R_{14} + H^2 L_1^2 + R_{13}^2.$

The 2nd order principal minor is positive if $\frac{2}{q^2}\Omega_1 > \Omega_2$.

At the stationary point, the 3rd order principal minor is as follows:

$$| H_{33} | = \begin{vmatrix} J_{qq} & J_{ql} & J_{qS} \\ J_{lq} & J_{ll} & J_{lS} \\ J_{Sq} & J_{Sl} & J_{SS} \end{vmatrix} = \frac{1}{P^2 q^3 \alpha Q} \left[\Omega_4 - \Omega_3 \right]$$

where, $\Omega_3 = qPS_2 L_4 R_{12} \Big\{ 2PS_2 L_4 R_{12} \left(\Pi_1 R_9 + \alpha Q (R_3 - DR_{12}) \right) - qS_2 \alpha Q \left(HL_1 + R_{13} \right) \left(PR_{12} - qH_m \right)$
$\Omega_4 = 2qP \left(S_5 R_5 + S_2 \right) \Big\{ 2P \left(HL_5 q + R_{14} \right) \left(\Pi_1 R_9 + \alpha Q (R_3 - DR_{12}) \right)$
$-q^2 P \alpha Q \left(HL_1 + R_{13} \right)^2 \Big\} + S_2 \left(PR_{12} - qH_m \right) \Big\{ q^2 P \alpha Q S_2 L_4 R_{12} \left(HL_1 + R_{13} \right)$
$-q \alpha Q S_2 \left(qHL_5 + R_{14} \right) \left(PR_{12} - qH_m \right) \Big\}.$

The principal minor of 3rd order be negative if $\Omega_3 > \Omega_4$. At the stationary point, the 4th order principal minor is as follows:

$$| H_{44} | = \begin{vmatrix} J_{qq} & J_{ql} & J_{qS} & J_{q\rho} \\ J_{lq} & J_{ll} & J_{lS} & J_{l\rho} \\ J_{Sq} & J_{Sl} & J_{SS} & J_{S\rho} \\ J_{\rho q} & J_{\rho l} & J_{\rho S} & J_{\rho\rho} \end{vmatrix}$$
$$= -\left(\xi_1 \xi_2 \rho^{\xi_2 - 1} \right)^2 \left(1 + 2\frac{S_3}{S_2} R_5 \right) \left[\Omega_6 - \Omega_5 \right] + \left[\frac{\eta}{2}(\xi_2 - 1) - \frac{\left(\xi_1 \xi_2 \rho^{\xi_2 - 1} \right)^2}{S_2} \right] | H_3 |$$

where $\Omega_5 = \frac{2}{q^2} \left\{ DR_{12} - \left(\frac{\Pi_1}{\alpha Q} R_9 + R_3 \right) \right\} \left(qL_5 H + R_{14} - S_2 L_4^2 R_{12}^2 \right) + \left(L_1 H + R_{13} \right) \left\{ L_1 H + R_{13} - S_2 \left(-\frac{H_m}{P} + \frac{L_4}{q} R_{12} \right) R_{12} \right\}$
$\Omega_6 = \left(-\frac{H_m}{P} + \frac{L_4}{q} R_{12} \right) \left\{ \left(L_1 H + R_{13} \right) S_2 L_4 R_{12} - S_2 \left(-\frac{H_m}{P} + \frac{L_4}{q} R_{12} \right) \left(qL_5 H + R_{14} \right) \right\}$. Since $| H_{33} | < 0$, the 4th order principal minor is positive if $\frac{\eta}{2}(\xi_2 - 1) < \frac{\left(\xi_1 \xi_2 \rho^{\xi_2 - 1} \right)^2}{S_2}$ and $\Omega_5 < \Omega_6$. At the stationary point, the 5th order principal minor is as follows:

$$| H_{55} | = \begin{vmatrix} J_{qq} & J_{ql} & J_{qS} & J_{q\rho} & J_{q\pi_x} \\ J_{lq} & J_{ll} & J_{lS} & J_{l\rho} & J_{l\pi_x} \\ J_{Sq} & J_{Sl} & J_{SS} & J_{S\rho} & J_{S\pi_x} \\ J_{\rho q} & J_{\rho l} & J_{\rho S} & J_{\rho\rho} & J_{\rho\pi_x} \\ J_{\pi_x q} & J_{\pi_x l} & J_{\pi_x S} & J_{\pi_x \rho} & J_{\pi_x \pi_x} \end{vmatrix}$$
$$= \beta_0 R_4 \left[\frac{2}{\pi_0} D | H_{44} | + \Pi_2 \left(L_4 D \Omega_8 + S_2 \Omega_9 \right) - \left(\frac{D}{q^2 \rho^4} \Omega_7 + \xi_1 \xi_2 \rho^{\xi_2 - 1} \Omega_{10} \right) \right]$$

The 5th order principal minor is negative if $\frac{2}{\pi_0} D | H_{44} | + \Pi_2 \left(L_4 D \Omega_8 + S_2 \Omega_9 \right) < \frac{D}{q} \Omega_7 + \xi_1 \xi_2 \rho^{\xi_2 - 1} \Omega_{10}$ for some $\Omega_7, \Omega_8, \Omega_9$ where, the value of Ω_7 is
$\Omega_7 = \xi_1^2 \xi_2^2 \rho^{2\xi_2 - 2} \beta_0 \Pi_2 R_4 R_{15} \Big[L_4 \left(L_4 - 1 \right) S_2^2 R_{12}^2 - \left(qHL_5 + R_{14} \right) \left(S_2 + 2S_3 R_5 \right)$
$+ L_4 DR_{12} \Big\{ S_2 \Big(L_4 - 2 \Big) - 2S_3 R_5 \Big\} \Big]$
$- \xi_1^2 \xi_2^2 \rho^{2\xi_2 - 2} \beta_0 \Pi_2 R_4 R_{12} \Big[-(L_1 H + R_{13})(S_2 + 2S_5 R_5)$
$+ S_2^2 R_{12} R_{15} (L_4 - 1) - \frac{q}{q} R_{12} \{ S_2 (L_4 + 2) + 2S_3 R_5 \} \Big]$
$+ \xi_1^2 \xi_2^2 \rho^{2\xi_2 - 2} \beta_0 \Pi_2 R_4 \Big[-(L_1 H + R_{13}) L_4 (D + S_2 R_{12})$
$+ S_2 R_{15} (qL_5 H + R_{14} + DL_4 R_{12}) + \frac{D}{q} (qL_5 H + R_{14} - S_2 L_4 R_{12}^2)) \Big]$
$- \frac{\eta}{2} (\xi_2 - 2) \beta_0 \Pi_2 R_4 \Big[-(L_1 H + R_{13}) \{ S_2^2 L_4 R_{12} + 2L_4 D (S_2 + S_3 R_5) \}$

$+ S_2^2 R_{15}(qL_5H + R_{14} + L_4^2 DR_{12}) + \frac{D}{q}\{2(qL_5H + R_{14})(S_2 + S_3R_5)$
$- S_2^2 L_4^2 R_{12}^2\}\Big]$. The 6th order principal minor is

$$
\mid H_{77} \mid = \begin{vmatrix} J_{qq} & J_{ql} & J_{qS} & J_{q\rho} & J_{q\pi_x} & J_{q\phi} \\ J_{lq} & J_{ll} & J_{lS} & J_{l\rho} & J_{l\pi_x} & J_{l\phi} \\ J_{Sq} & J_{Sl} & J_{SS} & J_{S\rho} & J_{S\pi_x} & J_{S\phi} \\ J_{\rho q} & J_{\rho l} & J_{\rho S} & J_{\rho\rho} & J_{\rho\pi_x} & J_{\rho\phi} \\ J_{\pi_x q} & J_{\pi_x l} & J_{\pi_x S} & J_{\pi_x\rho} & J_{\pi_x\pi_x} & J_{\pi_x\phi} \\ J_{\phi q} & J_{\phi l} & J_{\phi S} & J_{\phi\rho} & J_{\phi\pi_x} & J_{\phi\phi} \end{vmatrix}
$$

$$
= 2D\beta_0 \frac{a}{\phi}\Pi_2 R_4 \left\{ \left(\frac{S_2}{D}\Omega_{11} + \frac{1}{\pi_0\Pi_2}\Omega_{13} - \frac{1}{2\phi} \mid H_{55} \mid \right) - \frac{\xi_1\xi_2\rho^{\xi_2-1}}{D}\Omega_{12} \right\}
$$

for some $\Omega_{11}, \Omega_{12}, \Omega_{13}$. Since $H_{55} < 0$ therefore, if $\frac{S_2}{D}\Omega_{11} + \frac{1}{\pi_0\Pi_2}\Omega_{13} - \frac{1}{2\phi} \mid H_{55} \mid > \frac{\xi_1\xi_2\rho^{\xi_2-1}}{D}\Omega_{12}$ then the 6th order principal minor becomes positive.

The value of the 7th order principal minor $\mid H_{77} \mid$ is as follows:

$$
\mid H_{77} \mid = \begin{vmatrix} J_{qq} & J_{ql} & J_{qS} & J_{q\rho} & J_{q\pi_x} & J_{q\phi} & J_{qk} \\ J_{lq} & J_{ll} & J_{lS} & J_{l\rho} & J_{l\pi_x} & J_{l\phi} & J_{lk} \\ J_{Sq} & J_{Sl} & J_{SS} & J_{S\rho} & J_{S\pi_x} & J_{S\phi} & J_{Sk} \\ J_{\rho q} & J_{\rho l} & J_{\rho S} & J_{\rho\rho} & J_{\rho\pi_x} & J_{\rho\phi} & J_{\rho k} \\ J_{\pi_x q} & J_{\pi_x l} & J_{\pi_x S} & J_{\pi_x\rho} & J_{\pi_x\pi_x} & J_{\pi_x\phi} & J_{\pi_x k} \\ J_{\phi q} & J_{\phi l} & J_{\phi S} & J_{\phi\rho} & J_{\phi\pi_x} & J_{\phi\phi} & J_{\phi k} \\ J_{kq} & J_{kl} & J_{kS} & J_{k\rho} & J_{k\pi_x} & J_{k\phi} & J_{kk} \end{vmatrix}
$$

$$
= \frac{\Pi_1}{q} R_{10}\Omega_{14} - L_4\Pi_1 R_{10}\Omega_{15} - S_2\bar{\pi} R_4\Omega_{16} + \frac{\Pi_1}{\alpha Q} R_{11} \mid H_{66} \mid
$$

$$
= \Pi_1 \left(\frac{1}{q} R_{10}\Omega_{14} + \frac{1}{\alpha Q} R_{11} \mid H_{66} \mid - L_4 R_{10}\Omega_{15} \right) - S_2\bar{\pi} R_4\Omega_{16}
$$

for some $\Omega_{14}, \Omega_{15}, \Omega_{16}$, where, if $\Pi_1 \left(\frac{1}{q} R_{10}\Omega_{14} + \frac{1}{\alpha Q} R_{11} \mid H_{66} \mid - L_4 R_{10}\Omega_{15} \right) < S_2\bar{\pi} R_4\Omega_{16}$, then the principal minor of 7th order satisfies the property of negative definite.

The principal minor of 8th order $\mid H_{88} \mid$ is given by

$$
\mid H_{88} \mid = -\frac{D}{\alpha Q}\kappa \mid H_{77} \mid > 0
$$

since $\mid H_{77} \mid$ is negative.

The 9th order minor $\mid H_{99} \mid$ is given by

$$
\mid H_{99} \mid = -\frac{D}{\alpha Q}\mu \mid H_{88} \mid < 0
$$

since $\mid H_{88} \mid > 0$. Hence, the principal minor is satisfied the property of positive definite.

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